EXACT TESTS FOR FIXED AND RANDOM EFFECTS IN UNBALANCED LINEAR MIXED MODELS

By

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To my wife, Betty, and my children,

Gabriel and Natalia

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EXACT TESTS FOR FIXED AND RANDOM EFFECTS IN UNBALANCED LINEAR MIXED MODELS

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The testing of variance components in the case of unbalanced designs has relied on approximate techniques, particularly on Satterthwaite's approximation of the test statistic. While this procedure has performed well with some unbalanced designs, it has performed poorly in many others. Furthermore, the exact distributions of the approximate test statistics are unknown under both the null and alternative hypotheses. Hence, the appeal for exact testing techniques is understandable.

The problem of deriving exact tests for variance components in two unbalanced random-effects models has been solved in the special cases of the 2-fold nested and two-way cross classification models. Other exact tests have been proposed in the past for these random models, but with questionable assumptions.

Only two articles have explored the problem of the unbalanced mixed model is involved. Furthermore, their presentation leaves the problem of testing the fixed effects unanswered.

I present exact tests for testing the variance components and for testing the fixed effects in unbalanced 2-fold nested and two-way cross classification mixed models. In addition, I consider a conservative test for the variance component of the nesting effect in a random 2-fold nested model.

The exact tests are compared to Satterthwaite's approximate tests, with respect to power, using designs with different degrees of imbalance.

CHAPTER ONE
INTRODUCTION

The technique that we will illustrate is based on obtaining a random vector having a diagonal variance-covariance matrix. Once we obtain this vector we transform the model into a fixed-effects model and apply the theory of least squares. The 2-fold nested mixed model will be used to outline the derivations involved, followed by a brief discussion of the mixed two-way cross classification model.

Consider the 2-fold nested mixed model,

$$y_{ijk} = \mu + \alpha_i + b_{j(i)} + e_{ijk};$$

$$i=1,2,\ldots,a;\; j=1,2,\ldots,m_i;\; k=1,2,\ldots,n_{j(i)}, \qquad (1.1)$$

where μ is an unknown constant; α_i is a fixed unknown parameter; $b_{j(i)}$ and e_{ijk} are independently distributed as $N(0,\sigma_b^2)$ and $N(0,\sigma_e^2)$, respectively. The objective is to derive an exact test statistic for testing estimable linear functions of the fixed effects,

$$H_o\colon R'[\mu : \underset{\sim}{\alpha}']' = \underset{\sim}{m}, \qquad (1.2)$$

where $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_a)'$; R is a matrix of full row rank of independent estimable linear functions of $\underset{\sim}{\alpha}$ and $\underset{\sim}{m}$ is a vector of known constants. We begin by considering the residual sum of squares of the model, that is,

$$Q_\varepsilon = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2, \tag{1.3}$$

which we can express in matrix form as

$$Q_\varepsilon = \underset{\sim}{y}' W \underset{\sim}{y}, \tag{1.4}$$

by using $W = \left\{ I_{n_{..}} - \bigoplus_{i,j} \left(J_{n_{j(i)}} / n_{j(i)} \right) \right\}$ ($\oplus$ is the direct sum operator) and $\underset{\sim}{y}$ is the vector of observations. Now, the matrix W is an idempotent matrix of rank equal to $n_{..} - m_.$ $(m_. = \sum_{i=1}^{a} m_i)$; hence, there exists an orthogonal matrix, H, such that

$$W = H\Lambda H', \tag{1.5}$$

where Λ is a diagonal matrix whose diagonal elements are the eigenvalues of W. (Since W is idempotent its eigenvalues are 0 and 1.) Assuming that $n_{..} > 2m_.$ and by noticing the form of W in (1.4) we can partition H and Λ into

$$\begin{aligned} H &= \left[H_1 : H_2 : H_3 \right] \\ \Lambda &= \mathrm{Diag}\left[I_{m_.} : I_{n_{..} - 2m_.} : \underset{\sim}{0} \right] \end{aligned} \tag{1.6}$$

where H_1, H_2, H_3 are matrices of orders $n_{..} \times m_{.}$, $n_{..} \times (n_{..} - 2m_{.})$, and $n_{..} \times m_{.}$, respectively; $\underset{\sim}{0}$ is an $m_{.} \times m_{.}$ matrix of zeroes.

To obtain a random vector having a diagonal variance-covariance matrix, let us consider the vector $\underset{\sim}{w}$ defined as

$$\underset{\sim}{w} = \bar{\underset{\sim}{y}} + \left(\lambda_{\max} I_{m_{.}} - K\right)^{1/2} H_1' \underset{\sim}{y} , \tag{1.7}$$

where $\bar{\underset{\sim}{y}}$ is the vector of cell means, $\bar{y}_{ij.} = (1/n_{j(i)}) \sum_{k}^{n_{j(i)}} y_{ijk}$; $K = \mathrm{Diag}\left(n_{1(1)}^{-1}, n_{2(1)}^{-1}, \ldots, n_{m_a(a)}^{-1}\right)$; $\lambda_{\max}$ is the largest eigenvalue of K; the vector $\underset{\sim}{w}$ is well-defined and has a mean vector and a variance-covariance matrix given by

$$\begin{aligned} E(\underset{\sim}{w}) &= \mu \underset{\sim}{1}_{m_{.}} + B_1 \underset{\sim}{\alpha}, \\ \mathrm{Var}(\underset{\sim}{w}) &= I_{m_{.}} \left(\sigma_b^2 + \lambda_{\max} \sigma_e^2\right) \end{aligned} \tag{1.8}$$

where $B_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i}$. Now, let

$$\begin{aligned} X &= \left[\underset{\sim}{1}_{m_{.}} : B_1\right] \\ \underset{\sim}{\rho} &= \left[\mu : \underset{\sim}{\alpha}'\right] \\ \underset{\sim}{\varepsilon}^* &= I_{m_{.}} \underset{\sim}{b} + \bar{\underset{\sim}{\varepsilon}} + \left(\lambda_{\max} I_{m_{.}} - K\right)^{1/2} H_1' e, \end{aligned} \tag{1.9}$$

where $\underset{\sim}{b} = \left(b_{1(1)}, b_{2(1)}, \ldots, b_{m_a(a)}\right)$.

Hence, $\underset{\sim}{w}$ can be written as

$$\underset{\sim}{w} = X\underset{\sim}{\rho} + \underset{\sim}{\varepsilon}^{*} . \qquad (1.10)$$

This model has a residual sum of squares of

$$SSE = \underset{\sim}{w}' \left(I_{m_{\cdot}} - X(X'X)^{-}X'\right)\underset{\sim}{w}, \qquad (1.11)$$

which is distributed as a $\chi^2_{m_{\cdot}-a}$.

To obtain a test statistic that can be used to test the hypothesis (1.2), consider the quadratic form

$$Q_a^{(M)} = \underset{\sim}{\ell}_a'(R'G_xR)^{-1}\underset{\sim}{\ell}_a/\sigma^{*2},$$

$$\sim \chi^2[s,(R'\underset{\sim}{\rho}-\underset{\sim}{m})'(R'G_x\underset{\sim}{R})^{-1}(R'\underset{\sim}{\rho}-\underset{\sim}{m})/2\sigma^{*2}] \qquad (1.12)$$

where $\underset{\sim}{\ell}_a = R'\underset{\sim}{\rho}^{o} - \underset{\sim}{m}$, $\overset{o}{\underset{\sim}{\rho}}$ is a solution to the normal equations $(X'X)\underset{\sim}{\rho}^{o} = X'\underset{\sim}{w}$; $G_x = (X'X)^{-}$; $\sigma^{*2} = \sigma_b^2 + \lambda_{max}\sigma_e^2$; $s = \text{rank}(R')$. The quadratic form $Q_a^{(M)}$ and SSE (1.11) are independently distributed. Hence to test the hypothesis (1.2), we can use the following test statistic:

$$F(H) = \frac{Q_a^{(M)}/s}{SSE/(m_{\cdot}-a)} \sim F[s,(m_{\cdot}-a),(R'\underset{\sim}{\rho}-\underset{\sim}{m})'(R'G_xR)^{-1}(R'\underset{\sim}{\rho}-\underset{\sim}{m})/2\sigma^{*2}].$$

$$(1.13)$$

Note that under H_o: $R'\underset{\sim}{\rho} = \underset{\sim}{m}$, F(H) has a central F distribution with s and $m_{\bullet}-a$ numerator and denominator degrees of freedom, respectively. Hence, we have an exact test for the fixed effects.

In a similar fashion an exact test is derived for the fixed effects in a two-way cross classification mixed model. We again attempt to arrive at a random vector, as in (1.7); however, unlike the 2-fold nested mixed model, where $\text{Var}(\bar{\underset{\sim}{y}})$ involves only diagonal matrices $\left(\text{Var}(\bar{\underset{\sim}{y}}) = I_{m_{\bullet}}\sigma_b^2 + K\sigma_e^2\right)$, the $\text{Var}(\bar{\underset{\sim}{y}}^*)$ (where * denotes a two-way cross classification model) involves matrices which are not diagonal. Rather,

$$\text{Var}(\bar{\underset{\sim}{y}}^*) = A_2\sigma_\beta^2 + I_{rs}\sigma_{\alpha\beta}^2 + K^*\sigma_e^2 , \qquad (1.14)$$

where $A_2 = J_s \times I_r$ (x is the direct product operator) and $K^* = \text{Diag}\left(n_{11}^{-1}, n_{12}^{-1}, \ldots, n_{rs}^{-1}\right)$. Note that the matrix A_2 is not a diagonal matrix. Furthermore, note that in model (1.1), we have only one set of random effects aside from random error; hence, only one variance component exists in addition to the variance component associated with the error; whereas, in a two-way cross classification mixed model, we have two sets of random effects--a random effect and a random interaction effect, with variance components σ_β^2 and $\sigma_{\alpha\beta}^2$, respectively. In order to proceed as above, we need a transformation which will consequently allow us to form a ratio consisting of two quadratic forms, depending on the same set of variance components, with the denominator distributed as a central chi-square under both the null and alternative hypotheses.

Consider the two-way cross classification mixed model:

$$y^*_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + e_{ijk};$$

$$i=1,2,\ldots,r;\ j=1,2,\ldots,s;\ k=1,2,\ldots,n_{ij}, \qquad (1.15)$$

where μ is an unknown constant; α_i is an unknown fixed parameter; β_j, $(\alpha\beta)_{ij}$, and e_{ijk} are independently distributed as $N(0,\sigma^2_\beta)$, $N(0,\sigma^2_{\alpha\beta})$, and $N(0,\sigma^2_e)$, respectively.

Define $\bar{y}^*_{ij.} = (1/n_{ij}) \sum_k y_{ijk}$, so that in matrix notation

$$\bar{\underset{\sim}{y}}^* = \mu \underset{\sim}{1}_{rs} + B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + I_{rs}\alpha\beta + \bar{\underset{\sim}{e}}\ , \qquad (1.16)$$

where $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$; $\underset{\sim}{\beta} = (\beta_1, \beta_2, \ldots, \beta_s)'$; $\alpha\beta = (\alpha\beta_{11}, \alpha\beta_{12}, \ldots, \alpha\beta_{rs})'$; $B_1 = I_r \times \underset{\sim}{1}_s$ and $B_2 = \underset{\sim}{1}_r \times I_s$; $\bar{\underset{\sim}{e}} \sim N(\underset{\sim}{0}, K^*\sigma^2_e)$.

Consider the matrices $A_1 = B_1B_1'$ and $A_2 = B_2B_2'$. These matrices commute, hence, there exists an orthogonal matrix P that simultaneously diagonalizes both A_1 and A_2. Let

$$\underset{\sim}{u} = P\bar{\underset{\sim}{y}}^*. \qquad (1.17)$$

As we will see in Chapter Three, this random vector, $\underset{\sim}{u}$, can be partitioned into

$$
\begin{aligned}
&u_1 \sim N_1\left(\mu(rs)^{1/2} + (rs)^{-1/2}\underset{\sim}{1}'_{rs}B_1\underset{\sim}{\alpha}, r\sigma^2_\beta + \sigma^2_{\alpha\beta} + (rs)^{-1}[\textstyle\sum_{ij} n^{-1}_{ij}]\sigma^2_e\right),\\
&\underset{\sim}{u}_2 \sim N_{r-1}\left(P_2B_1\underset{\sim}{\alpha}, \sigma^2_{\alpha\beta}I_{r-1} + P_2K^*P'_2\sigma^2_e\right),\\
&\underset{\sim}{u}_3 \sim N_{s-1}\left(\underset{\sim}{0}, (r\sigma^2_\beta + \sigma^2_{\alpha\beta})I_{s-1} + P_3K^*P'_3\sigma^2_e\right),\\
&\underset{\sim}{u}_4 \sim N_{(r-1)(s-1)}\left(\underset{\sim}{0}, \sigma^2_{\alpha\beta}I_{(r-1)(s-1)} + \underset{\sim}{P}_4K^*P'_4\sigma^2_e\right),
\end{aligned}
\tag{1.18}
$$

where P_2, P_3, P_4 are the corresponding submatrices of P (see CHAPTER THREE, Section 3.4.2). What is important here is to note that the variance-covariance matrices for $\underset{\sim}{u}_2$ and $\underset{\sim}{u}_4$ depend on the same set of variance components; hence, we can consider concatenating these two random vectors into

$$
\underset{\sim}{u}_{2:4} = [\underset{\sim}{u}'_2 : \underset{\sim}{u}'_4]'. \tag{1.19}
$$

From the residual sum of squares associated with the model we can obtain a partition similar to (1.6) and define the random vector $\underset{\sim}{w}$ as

$$
\underset{\sim}{w} = \underset{\sim}{u}_{2:4} + (\lambda^F_{max}I_{s(r-1)} - L_{2:4})^{1/2}H'_{F1}\underset{\sim}{y}^*. \tag{1.20}
$$

where $$L_{2:4} = \begin{bmatrix} P_2K^*P'_2 & P_2K^*P'_4 \\ P_4K^*P'_2 & P_4K^*P'_4 \end{bmatrix};$$

$\lambda^F_{\max}$ is the largest eigenvalue of $L_{2:4}$; H_{F1} is derived similarily to the H_1 matrix in (1.6); $\underset{\sim}{y}^*$ is the vector of observations. Following the steps in (1.10), (1.11), and (1.12) we derive an exact test concerning estimable linear functions of $\underset{\sim}{\alpha}$ in (1.16), similar to the exact test derived for the fixed effects in the 2-fold nested mixed model (see (1.13)).

In addition to the tests for the fixed effects, exact tests are derived for the variance components associated with the random effects in both models (1.1) and (1.15). The development of an exact test for the variance component associated with the nested effect in the 2-fold nested model essentially involves deriving a random vector having expectation $\underset{\sim}{0}$. This involves pre-multiplying the vector of cell means by a matrix, C, whose rows define orthogonal contrasts so that

$$CE(\bar{\underset{\sim}{y}}) = \underset{\sim}{0} \ , \tag{1.21}$$

The derivation of the exact tests for the variance components associated with the random effects in the two-way cross classification mixed model involves the technique outlined above for testing the fixed effects.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

As articulated in Cummings and Gaylor (1974), the testing of variance components when we have an unbalanced design, in general, cannot employ the test statistics derived for the analogous balanced situation. Consider the random-effects 2-fold nested model with variance components σ_a^2, σ_b^2, and σ_e^2, associated with the nesting, nested, and error factors, respectively. The hypothesis H_o: $\tau_b = 0$ (where $\tau_b = \sigma_b^2/\sigma_e^2$) can be tested using an exact test based on the ratio of the mean square associated with the nested effect (MSβ) to the error mean square. This same test is employed when the data set is balanced. On the other hand, we know that in general, in an unbalanced data situation the mean square associated with the nesting effect (MSα) does not have a chi-square type distribution. Furthermore, this mean square in general is not independently distributed from MSβ. Hence, we cannot employ the analogous test statistic derived for balanced data to test the hypothesis H_{o_1}: $\tau_a = 0$ (where $\tau_a = \sigma_a^2/\sigma_e^2$), that is, MSα/MSβ. Cummings and Gaylor noted that the expected value of MSα and MSβ are not, in general, equal under H_{o_1}.

To test H_{o1}, Cummings and Gaylor used Satterthwaite's (1946) approximation. They derived a ratio whose denominator is a linear combination of the nested and error mean squares, such that, its expected value is equivalent to the expected value of $MS\alpha$ under H_{o1}. Satterthwaite's approximation requires the mean squares involved to be independently distributed with chi-square type distributions. As noted above, these assumptions may be violated when we have unbalanced data.

Cummings and Gaylor studied the effects of failures of these assumptions under Satterthwaite's approximation. They found that when $MS\alpha$ and $MS\beta$ were independently distributed but did not have chi-square type distributions, the actual test size was equal to the stated test size when $\tau_b = 0$; however, the computed test size was increasingly larger than the stated test size with increasing values of τ_b due to the increasing nonchi-squaredness of $MS\alpha$ and $MS\beta$. When they considered dependent $MS\alpha$'s and $MS\beta$'s having chi-square type distributions, the opposite occurred, namely, with increasing values of τ_b, the computed test size was increasingly smaller than the stated test size. When both violations occur, that is, a non-chi-squaredness and a dependency of the mean squares, the effects appear to cancel each other and the stated test size is close to the estimated test size.

As pointed out in Tietjen (1974), there are conditions for unbalanced 2-fold nested designs where the assumptions of chi-square and independency are not violated. For instance, when we have last-stage uniformity, that is $n = n_{j(i)}\ \forall_{ij}$, MSα and MSβ are independent and distributed as scaled chi-square variates. In addition, E(MSα) and E(MSβ) are equal under H_o, hence, the conventional F-test,

$$F = MS\alpha/MS\beta, \tag{2.1}$$

provides an exact test for testing H_{o_1}.

In the following section, we will briefly discuss past and recent developments concerning the exact testing of variance components when we have unbalanced data.

2.2 Exact Tests

Harville and Fenech (1985) considered the derivation of an exact confidence interval for the ratio of the variance component associated with a set of random effects and the variance component associated with the error. The derivation is based on an unbalanced mixed model with only a single set of random effects (an extension is suggested in later chapters of the paper). Consider the mixed model ($\underset{\sim}{y}$ is n×1 random vector):

$$\underset{\sim}{y} = X\underset{\sim}{\alpha} + Z\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}, \tag{2.2}$$

where $\underset{\sim}{\alpha}$ is a p×1 vector of fixed effects; $\underset{\sim}{\beta}$ and $\underset{\sim}{\varepsilon}$ are independently distributed random vectors of orders m × 1 and n×1, respectively, with $\underset{\sim}{\beta} \sim \text{MVN}(\underset{\sim}{0}, \sigma_\beta^2 I)$ and $\underset{\sim}{\varepsilon} \sim \text{MVN}(\underset{\sim}{0}, \sigma_\varepsilon^2 I)$; X and Z are given matrices. The objective is to derive an exact confidence interval for $\tau_\beta = \sigma_\beta^2/\sigma_\varepsilon^2$.

Essentially, Harville and Fenech consider treating the effects in the model as if they were all fixed. They arrive at the normal equations:

$$\begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{\alpha}} \\ \hat{\underset{\sim}{\beta}} \end{bmatrix} = \begin{bmatrix} X'\underset{\sim}{y} \\ Z'\underset{\sim}{y} \end{bmatrix} . \qquad (2.3)$$

From (2.3), the reduced normal equations (see Searle (1971) p. 264) are obtained,

$$C_H \hat{\underset{\sim}{\beta}} = \underset{\sim}{q}, \qquad (2.4)$$

where

$$\begin{aligned} C_H &= Z'\left(I - X'(X'X)^{-}X\right)Z; \\ \underset{\sim}{q} &= Z'\left(I - X'(X'X)^{-}X\right)\underset{\sim}{y}. \end{aligned} \qquad (2.5)$$

Let $r = \text{rank}(X,Z) - \text{rank}(X)$ and $f = n - \text{rank}(X,Z)$. Now the matrix C_H is a positive semidefinite matrix of rank r; hence, C_H has r nonzero eigenvalues $(\Delta_1, \Delta_2, \ldots, \Delta_r)$ and m-r eigenvalues equal to zero. Let R_H and U_H denote matrices whose columns are the orthonormal eigenvectors corresponding to the positive and zero eigenvalues of C_H, respectively. Let $D_H = \text{Diag}(\Delta_1, \Delta_2, \ldots, \Delta_r)$.

From Harville and Fenech (1985, p.140),

$$\underset{\sim}{t} = (t_1, t_2, \ldots, t_r) = D_H^{1/2} R_H' \underset{\sim}{\beta} = D_H^{-1/2} R_H' C_H \underset{\sim}{\beta} \qquad (2.6)$$

is an r×1 vector of linearly independent estimable functions of $\underset{\sim}{\beta}$. Furthermore, the best linear unbiased estimator (BLUE) of $\underset{\sim}{t}$ (treating $\underset{\sim}{\beta}$ as a vector of unknown parameters) is

$$\hat{\underset{\sim}{t}} = (\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_r) = D_H^{-1/2} R_H' \underset{\sim}{g} \qquad (2.7)$$

with

$$\begin{aligned} E(\hat{\underset{\sim}{t}}) &= \underset{\sim}{0} \\ \operatorname{Var}(\hat{\underset{\sim}{t}}) &= \sigma_\varepsilon^2 \operatorname{Diag}(1 + \tau_\beta \Delta_1, \ldots, 1 + \tau_\beta \Delta_r). \end{aligned} \qquad (2.8)$$

Now consider,

$$G(\tau_\beta, \underset{\sim}{y}) = (f/r) \sum_{i=1}^{r} t_i^2/(1+\tau_\beta \Delta_i)/\mathrm{SSE} \qquad \left(-\frac{1}{\Delta^*} < \tau_\beta < \infty\right), \qquad (2.9)$$

where SSE = error sum of squares = $\underset{\sim}{y}'(I - X'(X'X)^- X)\underset{\sim}{y} - \hat{\underset{\sim}{\beta}}'\underset{\sim}{g}$; $\Delta^* = \max(\Delta_1, \Delta_2, \ldots, \Delta_r)$.

We have that $G(\tau_\beta, \underset{\sim}{y}) \sim F(r,f)$ for $-\gamma^{*-1} < \tau_\beta < \infty$, where γ^* is the largest eigenvalue of Z´Z. Furthermore, $G(\tau_\beta, \underset{\sim}{y})$ is a strictly decreasing function of τ_β; hence, if τ_1 and τ_2 are defined as

$$P\left(G(\tau_1, \underset{\sim}{y}) > F_{\alpha_1}(r,f)\right) = \alpha_1$$

and (2.10)

$$P\left(G(\tau_2, \underset{\sim}{y}) > F_{1-\alpha_2}(r,f)\right) = 1-\alpha_2,$$

where $F_\alpha(r,f)$ is the upper α-100th percentile point of the F distribution with r and f degrees of freedom, then $\tau_1 < \tau_\beta < \tau_2$ is a $100(1-\alpha)\%$ confidence interval for τ_β (where $\alpha = \alpha_1 + \alpha_2$).

To test the hypothesis H_o: $\tau_\beta = \tau_o$ vs H_a: $\tau_\beta > \tau_o$ at the α^*-level, where $-1/\gamma^* < \tau_o < \infty$, we find the value of α for which $G(\tau_o, \underset{\sim}{y}) = F_\alpha$, i.e., the p-value of the test. We reject Ho when $\alpha < \alpha^*$.

The intellection of deriving a quadratic form which is a monotonic function of the parameter being tested is not a new concept in variance component inference techniques. For example, Wald's (1940), was the first paper to contemplate the testing of variance components when the data are unbalanced. In this paper, Wald studied a simple one-way classification model:

$$y_{ij} = \eta_i + \varepsilon_{ij};$$

$$i=1,2,\ldots a;\ j=1,2,\ldots m_i, \qquad (2.11)$$

where $\eta_i \sim N(\mu,\sigma_\eta^2)$; $\varepsilon_{ij} \sim N(0, \sigma^2)$; η_i and ε_{ij} are independently distributed $\forall_{ij}$. With the model in (2.11), the objective is to derive a test for the intraclass correlation parameter given by

$$\rho = \frac{\sigma^2_\eta}{\sigma^2_\eta + \sigma^2} = \frac{1}{1 + (\sigma^2_\eta/\sigma^2)} = \frac{1}{1 + \tau_\eta}, \tag{2.12}$$

where $\tau_\eta = \sigma^2_\eta/\sigma^2$. Since ρ is a monotonically decreasing function for increasing values of τ_η and the consequent test derived is a monotonic function of τ_η, Wald derived a test based on τ_η rather than ρ.

Consider the mean of the i^{th} group, $\bar{y}_{i.}$. Then,

$$E(\bar{y}_{i.}) = \mu \text{ and } Var(\bar{y}_{i.}) = \left(\frac{1}{m_i} + \tau_\eta\right)\sigma^2 \tag{2.13}$$

Let

$$Q_{WA} = \frac{1}{\sigma^2} \sum_{i=1}^{a} [w_i(\bar{y}_{i.} - \frac{\Sigma w_i \bar{y}_i}{\Sigma w_i})^2] \tag{2.14}$$

where $w_i = m_i/(1+m_i\tau_\eta)$. As shown in Wald (1940), $Q_{WA} \sim \chi^2_{a-1}$ and is independently distributed from

$$Q_{W\varepsilon} = \frac{\Sigma_i\Sigma_j(y_{ij} - \bar{y}_{i.})^2}{\sigma^2} \sim \chi^2_{n_{..}-a}. \tag{2.15}$$

Hence,

$$F(\tau_o) = \frac{Q_{WA}/(a-1)}{Q_{w\varepsilon}/(n_{..}-a)} \sim F(a-1,\ n_{..}-a). \qquad (2.16)$$

Wald showed that Q_{WA} is a monotonically decreasing function of τ_η. Consequently, we can test the hypothesis Ho: $\tau_\eta = \tau_o$ vs H_a: $\tau_\eta > \tau_o$. We reject Ho when $F(\tau_o)$ is "large," i.e., when $F(\tau_o)$ exceeds the upper α-percentile point of the F distribution with a-1 and $n_{..}$-a degrees of freedom in the numerator and denominator respectively.

In his 1941 paper, Wald extends his previous derivations in the one-way model to a random-effects model having more than one classification. Finally, in Wald (1947), the first paper dealing with the mixed model appeared and like his previous two papers, Wald derives a quadratic form that is a monotonic function of the parameter of interest. He approaches the problem in the framework of a regression analysis situation.

Spjøtvoll (1968) showed that Wald's (1947) method cannot be applied in general to any multiple classification model. Spjøtvoll considered a modification to Wald's method which he illustrated on the two-way classification model. He derived exact tests for the variance components associated with the interaction and main effects. However, his tests for the main effects depend on the assumption that the variance component of the interaction term is zero.

Seely and El-Bassiouni (1983) presented conditions under which Wald's (1947) method could be applied to mixed linear models. Developing Wald's method using reduction sum of squares principles, Seely and El-Bassiouni argue that Wald's method is applicable whenever r (= rank[X:Z] - rank(X)) and f (= $n_{..}$ - rank[X:Z]) are positive. The derivations in Harville and Fenech (1985) are based on these conditions.

Another significant technique for deriving an exact test was introduced by Thomsen (1975). Here the idea is to transfer the model to a "semi-canonical" form by using orthogonal transformations as we shall now show.

Consider the random two-way cross classification model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + e_{ijk};$$

$$i=1,2,\ldots,r;\ j=1,2,\ldots,s;\ k=1,2,\ldots,n_{ij}, \qquad (2.25)$$

where μ is an unknown constant parameter; α_i, β_j, $\alpha\beta_{ij}$, and e_{ijk} are independently distributed as $N(0,\sigma^2_\alpha)$, $N(0,\sigma^2_\beta)$, $N(0,\sigma^2_{\alpha\beta})$, and $N(0, \sigma^2_e)$, respectively. Let

$$\bar{y}_{ij.} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \bar{e}_{ij.}, \qquad (2.26)$$

where $\bar{e}_{ij.} = (\frac{1}{n_{ij}}) \sum_k e_{ijk}$. In matrix notation (2.26) becomes

$$\bar{\underset{\sim}{y}} = \mu 1_{rs} + B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + I\alpha\beta + \underset{\sim}{e}, \qquad (2.27)$$

where $B_1 = I_r \times \underset{\sim}{1}_s$ and $B_2 = \underset{\sim}{1}_r \times I_s$ (A×B signifies the direct product of the matrices A and B), and

$$\mathrm{Var}(\bar{\underset{\sim}{y}}) = A_1\sigma_\alpha^2 + A_2\sigma_\beta^2 + I\sigma_{\alpha\beta}^2 + K\sigma_e^2, \tag{2.28}$$

where $A_1 = B_1B_1'$ and $A_2 = B_2B_2'$; $K = \mathrm{Diag}\left(n_{11}^{-1}, n_{12}^{-1}, \ldots, n_{rs}^{-1}\right)$. Since A_1 and A_2 commute, there exists an orthogonal matrix P such that $PA_1P' = D_1$ and $PA_2P' = D_2$, where D_1 and D_2 are diagonal matrices whose diagonal elements are the eigenvalues of A_1 and A_2, respectively.

Thomsen considers the orthogonal transformation

$$\underset{\sim}{z} = P\bar{\underset{\sim}{y}} \tag{2.29}$$

(Note that $\mathrm{Var}(\underset{\sim}{z}) = PA_1P'\sigma_\alpha^2 + PA_2P'\sigma_\beta^2 + I_{rs}\sigma_{\alpha\beta}^2 + PKP'\sigma_e^2$.) The vector $\underset{\sim}{z}$ is partitioned into z_1, $\underset{\sim}{z}_\alpha$, $\underset{\sim}{z}_\beta$, and $\underset{\sim}{z}_{\alpha\beta}$, where

$$z_1 = (rs)^{-1/2}\underset{\sim}{1}_{rs}'\,\bar{\underset{\sim}{y}},$$

$$\underset{\sim}{z}_\alpha \sim N_{r-1}\left(\underset{\sim}{0}, sI\sigma_\alpha^2 + I\sigma_{\alpha\beta}^2 + K_1\sigma_e^2\right),$$

$$\underset{\sim}{z}_\beta \sim N_{s-1}\left(\underset{\sim}{0}, rI\sigma_\beta^2 + I\sigma_{\alpha\beta}^2 + K_2\sigma_e^2\right), \tag{2.30}$$

$$\underset{\sim}{z}_{\alpha\beta} \sim N_{(r-1)(s-1)}\left(\underset{\sim}{0}, I\sigma_{\alpha\beta}^2 + K_3\sigma_e^2\right),$$

where K_1, K_2, and K_3 are the corresponding submatrices of PKP'. From these random vectors Thomsen derives quadratic forms that one can use to test the variance components of the two-way cross classification model. However, as in Spjøtvoll (1968) the testing of the hypotheses $H_{o_1} : \sigma^2_\alpha/\sigma^2_e$ and $H_{o_1} : \sigma^2_\beta/\sigma^2_e$, requires the assumption that $\sigma^2_{\alpha\beta}/\sigma^2_e = 0$.

Khuri (1986b) approaches the problem in much the same way as Thomsen did but applies an additional step. The error sum of squares of the model can be written as

$$Q_{Ke} = \underset{\sim}{y}' R \underset{\sim}{y}, \tag{2.31}$$

where $\underset{\sim}{y}$ is the vector of observations and

$$R = I_{n_{\cdot\cdot}} - \underset{i,j}{\oplus} \left(J_{n_{ij}} / n_{ij} \right) \tag{2.32}$$

is an idempotent matrix of rank = $n_{\cdot\cdot}$-rs ($\oplus$ is the direct sum operator). Under the assumption that $n_{\cdot\cdot} > 2rs - 1$, it can be shown that Q_{Ke} can be partitioned as

$$Q_{Ke} = \underset{\sim}{y}' C_1' C_1 \underset{\sim}{y} + \underset{\sim}{y}' C_2' C_2 \underset{\sim}{y}$$

$$= Q_{1e} + Q_{2e}\ , \tag{2.33}$$

where $Q_{1e}/\sigma_e^2 = \underset{\sim}{y}'C_1'C_1\underset{\sim}{y}/\sigma_e^2$ and $Q_{2e}/\sigma_e^2 = \underset{\sim}{y}'C_2'C_2\underset{\sim}{y}/\sigma_e^2$ are independently distributed chi-squares having degrees of freedom $rs-1$ and $n_{..}-2rs+1$, respectively; C_2 is a matrix of order $n_{..}\times(n_{..}-2rs+1)$ such that $C_2'C_2 = I_{n_{..}-2rs+1}$ and $C_1'C_2 = \underset{\sim}{0}$.

Using the random vector $\underset{\sim}{w}$ defined as

$$\underset{\sim}{w} = \underset{\sim}{u} + \left(\lambda_m I_{rs-1} - L\right)^{1/2} C_1'\underset{\sim}{y}, \tag{2.34}$$

where $\underset{\sim}{u} = \left(\underset{\sim}{z}_\alpha', \underset{\sim}{z}_\beta', \underset{\sim}{z}_{\alpha\beta}'\right)'$ and L is a submatrix of PKP' which corresponds to $\underset{\sim}{\mu}$ (recall from above that P is the matrix that simultaneously diagonologizes both A_1 and A_2 and $K = \text{diag}(n_{11}^{-1}, n_{12}^{-1}, \ldots, n_{rs}^{-1})$), λ_m is the largest eigenvalue of L. Khuri derived four sums of squares which are distributed independently as scalar multiples of chi-squared random variables. These sums of squares were utilized to derive exact tests concerning the variance components σ_α^2, σ_β^2, and $\sigma_{\alpha\beta}^2$.

In spite of the arguments made by Thomsen (1975) and Spjøtvoll (1968) that the interaction variance component $\sigma_{\alpha\beta}^2$ must be zero, the variance components of the main effects can be tested without the assumption that $\sigma_{\alpha\beta}^2/\sigma_e^2 = 0$. However, the exact test developed for testing the hypothesis H_o: $\sigma_{\alpha\beta}^2/\sigma_e^2 = 0$ has fewer denominator degrees of freedom than the corresponding test derived by Thomsen (1975); hence, Thomsen's test is recommended when considering this hypothesis.

In Khuri (1986a), an exact test is derived for testing the variance component associated with the nesting factor in an unbalanced 2-fold nested model using a similar approach.

The derivations in Thomsen (1975), Khuri (1986a), and Khuri and Littell (1986b) are employed in the next chapter to derive exact tests for the random and fixed effects in mixed models.

CHAPTER THREE
DERIVATIONS OF EXACT TESTS

3.1 The Unbalanced 2-Fold Nested Random Effects Model

3.1.1 Introduction

Consider the following model:

$$y_{ijk} = \mu + \alpha_i + b_{j(i)} + e_{ijk};$$

$$i=1,2,\ldots,a;\ j=1,2,\ldots,m_i;\ k=1,2,\ldots,n_{j(i)}, \qquad (3.1)$$

where μ is an unknown constant parameter; α_i, $b_{j(i)}$, and e_{ijk} are independently normally distributed random variables with zero means and variances

σ_a^2, σ_b^2, and σ_e^2, respectively. Define

$$\bar{y}_{ij} = \left(1/n_{j(i)}\right) \sum_{k=1}^{n_{j(i)}} y_{ijk},$$

for $i=1,2,\ldots,a;\ j=1,2,\ldots,m_i$. From (3.1) we have

$$\bar{y}_{ij} = \mu + \alpha_i + b_{j(i)} + \bar{e}_{ij} , \tag{3.2}$$

where $\bar{e}_{ij} = (1/n_{j(i)}) \sum_{k=1}^{n_{j(i)}} e_{ijk}$. Alternatively, (3.2) can be written in matrix form as

$$\underset{\sim}{\bar{y}} = \mu \underset{\sim}{1}_m + B_1 \underset{\sim}{\alpha} + B_2 \underset{\sim}{\beta} + \underset{\sim}{\bar{e}}, \tag{3.3}$$

where $m = \sum_{i=1}^{a} m_i$; $\underset{\sim}{1}_m$ is an $m \times 1$ column vector of ones;

$B_1 = \underset{\sim}{1}_{m_1} \oplus \underset{\sim}{1}_{m_2} \oplus \ldots \oplus \underset{\sim}{1}_{m_a} = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i}$ ($A \oplus B$ signifies the direct sum of

the matrices A and B); $B_2 = \bigoplus_{i=1}^{m_i} I_{m_i} = I_m$; $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are column vectors, whose elements are α_i for $i=1,2,\ldots,a$, and $b_{j(i)}$ for $j=1,2,\ldots,m_i$, respectively.

Consider the variance-covariance matrix of $\underset{\sim}{\bar{y}}$:

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{\bar{y}}) &= B_1 B_1' \,\sigma_a^2 + I \,\sigma_b^2 + K \,\sigma_e^2 \\ &= A_1 \sigma_a^2 + I\sigma_b^2 + K\sigma_e^2, \end{aligned} \tag{3.4}$$

where $A_1 = B_1 B_1'$ and $K = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{m_i} n_{j(i)}^{-1}$. Now, since A_1 is a symmetric matrix, there exists an orthogonal matrix that diagonolizes A_1 (see Graybill 1983, p. 48). Let P be the orthogonal matrix with the property that $PA_1P' = D_1$, where D_1 is a diagonal matrix whose diagonal elements are the eigenvalues of A_1.

Lemma 3.1: The rank of D_1 is a.

Proof: $\text{rank}(D_1) = \text{rank}(PA_1P')$

$$\begin{aligned} &= \text{rank}(A_1) \text{ (see Searle 1982, p.197)} \\ &= \text{rank}(B_1) = \text{rank}\left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i} \right) \\ &= \sum_{i=1}^{a} \text{rank}(\underset{\sim}{1}_{m_i}) = a. \qquad \square \end{aligned}$$

The last step follows from Graybill (1983 p.229).

Now, since D_1 is an $m_{\bullet} \times m_{\bullet}$ diagonal matrix, it follows that D_1 has "a" nonzero diagonal elements and $m_{\bullet} - a$ zero diagonal elements. The matrix A_1 is nonnegative definite, hence, the nonzero diagonal elements of D_1 are positive.

Let $\underset{\sim}{z} = P\bar{\underset{\sim}{y}}$. From (3.4) we have

$$\begin{aligned} \text{Var}(\underset{\sim}{z}) &= P\{\text{Var}(\bar{\underset{\sim}{y}})\}P' \qquad (3.5) \\ &= PA_1P'\sigma_a^2 + I_{m_{\bullet}}\sigma_b^2 + PKP'\sigma_e^2 \\ &= D_1\sigma_a^2 + I_{m_{\bullet}}\sigma_b^2 + PKP'\sigma_e^2. \end{aligned}$$

We know that $m_1, m_2, \ldots, m_a$ are the "a" positive eigenvalues of the matrix A_1 (See Appendix A for proof.). Hence, from (3.5), we can express the variance-covariance matrix for $\underset{\sim}{z}$ as

$$\text{Var}(\underset{\sim}{z}) = \text{Diag}(m_1, m_2, \cdots, m_a, 0, 0, \cdots, 0)\,\sigma_a^2 + I_{m_{\bullet}}\sigma_b^2 + PKP'\sigma_e^2. \qquad (3.6)$$

In addition to determining the variance-covariance structure of $\underset{\sim}{z}$, we need to obtain $E(\underset{\sim}{z})$, where $E(\underset{\sim}{z})$ denotes the expected value of the vector $\underset{\sim}{z}$.

Since the positive eigenvalues of A_1 are $m_1, m_2, \ldots, m_a$ and $A_1 = \bigoplus_{i=1}^{a} J_{m_i}$, it follows that $m_i^{-1/2}\underset{\sim}{1}^{(i)}$ for $i=1,2,\ldots,a$, are the eigenvectors of associated with the positive eigenvalues $m_1, m_2, \ldots, m_a$ (See Appendix B for proof.), where

$$\underset{\sim}{1}^{(1)\prime} = [\underset{\sim}{1}'_{m_1} : \underset{\sim}{0}'_{1\times(m_{\bullet}-m_1)}]$$

$$\underset{\sim}{1}^{(2)\prime} = [\underset{\sim}{0}'_{m_1} : \underset{\sim}{1}'_{m_2} : \underset{\sim}{0}'_{1\times(m_{\bullet}-(m_1+m_2))}]$$

$$\vdots$$

$$\underset{\sim}{1}^{(a)\prime} = [\underset{\sim}{0}'_{1\times(m_{\bullet}-m_a)} : \underset{\sim}{1}'_{m_a}].$$

The matrix A_1 also has a zero eigenvalue with multiplicity $m_{\bullet}-a$. Let $\underset{\sim}{e}_{a+1}, \underset{\sim}{e}_{a+2}, \ldots, \underset{\sim}{e}_{m_{\bullet}}$ be the orthonormal eigenvectors associated with this eigenvalue. We thus have

$$
\begin{aligned}
\underset{\sim}{e}'_{a+1}\underset{\sim}{1}^{(i)} &= 0 \qquad (3.7)\\
\underset{\sim}{e}'_{a+2}\underset{\sim}{1}^{(i)} &= 0\\
&\vdots\\
\underset{\sim}{e}'_{m_{\cdot}}\underset{\sim}{1}^{(i)} &= 0,
\end{aligned}
$$

for $i=1,2,\ldots,a$. Note that from (3.7),

$$\underset{\sim}{e}'_{a+\ell}\underset{\sim}{1}_{m_{\cdot}} = 0, \qquad (3.8)$$

for $\ell=1,2,\ldots,(m_{\cdot}-a)$. The matrix P that diagonolizes A_1 can then be expressed as

$$P = \begin{bmatrix} m_1^{-1/2}\underset{\sim}{1}^{(1)'} \\ m_2^{-1/2}\underset{\sim}{1}^{(2)'} \\ \vdots \\ m_a^{-1/2}\underset{\sim}{1}^{(a)'} \\ \underset{\sim}{e}'_{a+1} \\ \vdots \\ \underset{\sim}{e}'_{m_{\cdot}} \end{bmatrix} \qquad (3.9)$$

From (3.8) and (3.9) it follows that

$$
\begin{aligned}
E(\underset{\sim}{z}) &= E(P\bar{\underset{\sim}{y}}) \\
&= PE(\bar{\underset{\sim}{y}}) \\
&= \mu P \underset{\sim}{1}_{m_\bullet} \\
&= \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{0} \end{bmatrix} ,
\end{aligned}
$$

where $\underset{\sim}{\mu}_1 = \mu(m_1^{1/2}, m_2^{1/2}, \ldots, m_a^{1/2})'$ of order $a \times 1$, and $\underset{\sim}{0}$ is an $(m_\bullet - a)\times 1$ vector of zeroes. Hence, we can partition the column vector $\underset{\sim}{z}$ into $\underset{\sim}{z}_a$ and $\underset{\sim}{z}_b$, where

$$
\begin{aligned}
\underset{\sim}{z}_a &\sim N_a\{\underset{\sim}{\mu}_1, [\bigoplus_{i=1}^{a} m_i \sigma_a^2 + I_a\sigma_b^2 + K^{(1)}\sigma_e^2]\} \\
\underset{\sim}{z}_b &\sim N_{m_\bullet - a}\{\underset{\sim}{0}_1, I_{m_\bullet - a}\sigma_b^2 + K^{(2)}\sigma_e^2]\}.
\end{aligned}
\tag{3.11}
$$

The matrices $K^{(1)}$ and $K^{(2)}$ are the corresponding submatrices of PKP'.

3.1.2 Derivation of an Exact Confidence Interval Concerning $\Delta_a = \sigma_a^2/\sigma_e^2$.

In this section we attempt to derive an exact confidence iterval for Δ_a by further analyzing the random vector $\underset{\sim}{z}_a$. Recall from (3.11) that

$$\underset{\sim}{z}_a \sim N_a\{\underset{\sim}{\mu}_1, [\bigoplus_{i=1}^{a} m_i\sigma_a^2 + I_a\sigma_b^2 + K^{(1)}\sigma_e^2]\}$$

where $\underset{\sim}{\mu}_1 = \mu(m_1^{1/2}, m_2^{1/2}, \ldots, m_a^{1/2})'$. The first objective will be to obtain a transformation on $\underset{\sim}{z}_a$ so that the resulting vector has zero expectation.

Consider the orthogonal matrix, $T = (\underset{\sim}{h}_1 : T_1)'$, where $\underset{\sim}{h}_1 = m_{\bullet}^{-1/2}(m_1^{1/2}, m_2^{1/2}, \ldots, m_a^{1/2} : \underset{\sim}{0})'$ and $\underset{\sim}{0}$ is an $m_{\bullet} - a$ column vector of zeroes. The matrix T_1 is a matrix of order $m_{\bullet} \times (m_{\bullet}-1)$ of rank $m_{\bullet}-1$, whose columns are obtained by applying the Gram-Schmidt orthonormalization (Golub (1969)) on the columns of the matrix

$$W_1 = I_{m_{\bullet}} - \underset{\sim}{h}_1\underset{\sim}{h}_1'. \qquad (3.12)$$

The Gram-Schmidt orthonormal factorization of the matrix W_1 is expressed as a product of two matrices, say U and V, where U is a column-orthonormal matrix of order $m_{\bullet} \times m_{\bullet}$ and V is an upper triangular matrix of order $m_{\bullet} \times m_{\bullet}$, such that $W_1 = UV$. Since, W_1 is an $m_{\bullet} \times m_{\bullet}$ matrix of rank equal to $m_{\bullet}-1$, one column of U is the zero vector and the corresponding row of V is also the zero vector. The matrix T_1 will consist of the matrix formed from the nonzero columns of U. These nonzero columns are orthonormal, hence, $T_1'T_1 = I_{m_{\bullet}-1}$. Furthermore, $T_1'\underset{\sim}{h}_1 = \underset{\sim}{0}_{(m_{\bullet}-1)\times 1}$, since $W_1\underset{\sim}{h}_1 = 0$ and the column space of T_1 is the same as the column space of W_1 (this follows from the Gram-Schmidt orthonormalization process). Obviously, $\mathbf{h}_1'\mathbf{h}_1 = 1$ and as mentioned above, the columns of T_1 are orthonormal, hence,

$$TT' = I_{m_{\cdot}} \tag{3.13}$$

where, $T = (\mathbf{h}_1 : T_1)'$. From Khuri (1985), the first row of the product TP is $m_{\cdot}^{-1/2}\mathbf{1}_{m_{\cdot}}'$. In addition, $TP(TP)' = I_{m_{\cdot}}$, hence, with the exception of the first row, each row of TP gives the coefficients for a contrast since they sum to zero.

Consider $\mathbf{u} = T_1'P\bar{\mathbf{y}}$. From the above results, we have

$$E(\mathbf{u}) = T_1'PE(\bar{\mathbf{y}}) = \mu T_1'P\mathbf{1}_{m_{\cdot}} = \mathbf{0} \tag{3.14}$$

and the variance-covariance matrix of $\mathbf{u}$ is

$$\begin{aligned} \mathrm{Var}(\mathbf{u}) &= T_1'PA_1P'T_1\sigma_a^2 + T_1'T_1\sigma_b^2 + T_1'PKP'T_1\sigma_e^2 \\ &= T_1'D_1T_1\sigma_a^2 + I_{m_{\cdot}-1}\sigma_b^2 + T_1'K^*T_1\sigma_e^2, \end{aligned} \tag{3.15}$$

where $K^* = PKP'$. The matrix $T_1'D_1T_1$ is symmetric, and therefore, there exists an orthogonal matrix L of order $(m_{\cdot}-1)\times(m_{\cdot}-1)$ such that $LT_1'D_1T_1L' = \Lambda_1$ is a diagonal matrix. Consider $\boldsymbol{\ell} = L\mathbf{u}$. It follows from (3.14) and (3.15) that

$$\begin{aligned} E(\underset{\sim}{\ell}) &= \underset{\sim}{0} \\ \mathrm{Var}(\underset{\sim}{\ell}) &= \Lambda_1\sigma_a^2 + I_{m_{\bullet}-1}\sigma_b^2 + LT_1'K^*T_1L'\sigma_e^2 \ . \end{aligned} \qquad (3.16)$$

The diagonal elements of Λ_1 are the nonnegative eigenvalues of $T_1'D_1T_1$. The number of positive diagonal elements is equal to the rank of $T_1'D_1T_1$, which is equal to a-1 (see Khuri (1985)).

Now consider the vector $\underset{\sim}{\ell}_a$ obtained from the elements of $\underset{\sim}{\ell}$ which correspond to the positive eigenvalues of $T_1'D_1T_1$. The vector $\underset{\sim}{\ell}_a$ has dimension a-1 with

$$\begin{aligned} E(\underset{\sim}{\ell}_a) &= \underset{\sim}{0} \\ \mathrm{Var}(\underset{\sim}{\ell}_a) &= \Lambda^*\sigma_a^2 + I_{a-1}\sigma_b^2 + K^{**}\sigma_e^2, \end{aligned} \qquad (3.17)$$

where Λ^* is a diagonal matrix whose diagonal elements are the positive eigenvalues of $T_1'D_1T_1$ and K^{**} is the corresponding submatrix of $LT_1'K^*T_1L'$. Now let

$$Q_{\underset{\sim}{\ell}_a} = \underset{\sim}{\ell}_a'(\Delta_a\Lambda^* + \Delta_bI_{a-1} + K^{**})^{-1}\underset{\sim}{\ell}_a/\sigma_e^2, \qquad (3.18)$$

where $\Delta_a = \sigma_a^2/\sigma_e^2$, $\Delta_b = \sigma_b^2/\sigma_e^2$.

Theorem 3.1

The quadratic form in (3.18) is distributed as

$$Q_{\ell_a} \sim \chi^2_{a-1} \; .$$

Proof: $\sigma_e^{-1/2}\left(\Delta_a \Lambda^* + \Delta_b I_{a-1} + K^{**}\right)^{-1}\left(\mathrm{Var}(\underset{\sim}{\ell}_a)\right) = I_{a-1}$, which is idempotent of rank a-1. Hence, from Searle (1971, p.58 Corollary 2.2), $Q_{\ell_a} \sim \chi^2(0)$ with a-1 degrees of freedom. □

Let $Q = \sum_{ijk} (y_{ijk} - \bar{y}_{ij\cdot})^2$. It is known that Q/σ_e^2 has a central chi-square distribution with $f = n_{\cdot\cdot} - m_\cdot$ degrees of freedom. Furthermore, since $Q_{\underset{\sim}{\ell}_a}$ depends on the normally distributed observations through a subset of the elements of the column vector of means, $\underset{\sim}{\bar{y}}$, it follows from Lindgren (1976, p.333, Theorem 3) that $Q_{\underset{\sim}{\ell}_a}$ and Q/σ_e^2 are independently distributed. Therefore,

$$F(\Delta_a, \Delta_b) = \frac{\left[\underset{\sim}{\ell}_a'(\Delta_a \Lambda^* + \Delta_b I_{a-1} + K^{**})^{-1} \underset{\sim}{\ell}_a\right]/(a-1)}{\sum_{ijk} (y_{ijk} - \bar{y}_{ij\cdot})^2/(n_{\cdot\cdot} - m_\cdot)} \tag{3.19}$$

has a central F with numerator and denominator degrees of freedom equal to a-1 and $n_{\cdot\cdot} - m_\cdot$, respectively.

We can now define a 100(1-α)% confidence region for (Δ_a, Δ_b) as

$$s(\underset{\sim}{y},\Delta_a,\Delta_b) = \{F(\Delta_a,\Delta_b) \leq F_{\alpha,a-1,n_{..}-m_{.}}\}, \tag{3.20}$$

where $F_{\alpha,a-1,n_{..}-m_{.}}$ is the upper α-percentile point of the corresponding F distribution. In order to obtain a confidence interval for Δ_a we can consider the projection of $s(\underset{\sim}{y},\Delta_a,\Delta_b)$ onto the Δ_a-axis. This will lead to a confidence interval for Δ_a with a confidence coefficient greater than or equal to $1-\alpha$.

We have obtained an exact but conservative confidence interval for Δ_a. This confidence interval can be used to test the hypothesis H_0: $\Delta_a = 0$. However, Khuri (1986a) has derived an exact test for Δ_a which is not conservative. It is recommended that his procedure be applied instead. One advantage of the exact confidence region for Δ_a and Δ_b derived is that it can be used to obtain simultaneous exact confidence intervals for all continuous functions of these parameters, similar to those obtained in Khuri (1981), for balanced random models.

3.2 The Unbalanced 2-Fold Nested Mixed Model

3.2.1 Introduction

Consider the following model:

$$y_{ijk} = \mu + \alpha_i + b_{j(i)} + e_{ijk};$$

$$i=1,2,\ldots,a;\ j=1,2,\ldots,m_i;\ k=1,2,\ldots,n_{j(i)}, \qquad (3.21)$$

where μ is an unknown constant parameter; α_i are the unknown fixed parameters; $b_{j(i)}$ and e_{ijk} are independent normally distributed random variables with zero means and variances σ_b^2 and σ_e^2, respectively. As with model

(3.1), define $\bar{y}_{ij.} = \left(1/n_{j(i)}\right) \sum_{k=1}^{n_{j(i)}} y_{ijk}$, for $i=1,2,\ldots,a$;

$j=1,2,\ldots,m_i$. It follows from (3.21) that

$$\bar{y}_{ij.} = \mu + \alpha_i + b_{j(i)} + \bar{e}_{ij.}, \qquad (3.22)$$

where $\bar{e}_{ij.} = \left(1/n_{j(i)}\right) \sum_{k=1}^{n_{j(i)}} e_{ijk}$. In matrix notation we have

$$\bar{\underset{\sim}{y}} = \mu \underset{\sim}{1}_{m.} + B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + \bar{\underset{\sim}{e}}, \qquad (3.23)$$

where $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, B_1, and B_2 are defined as in (3.3) (Recall from the last section that $B_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i}$ and $B_2 = I_{m_\bullet}$; $m_\bullet = \sum_{i=1}^{a} m_i$.).

3.2.2 Derivation of an Exact Test Concerning $\Delta_b = \sigma_b^2/\sigma_e^2$

Since, α_i for i=1,2,...,a are unknown fixed parameters, we have that the expected value of the vector $\bar{\underset{\sim}{y}}$ in (3.28) is

$$E(\bar{\underset{\sim}{y}}) = \mu \underset{\sim}{1}_{m_\bullet} + B_1\underset{\sim}{\alpha} \ . \tag{3.24}$$

Note that $B_1\underset{\sim}{\alpha} = \left(\alpha_1 \underset{\sim}{1}'_{m_1}, \alpha_2 \underset{\sim}{1}_{m_2}, \ldots, \alpha_a \underset{\sim}{1}'_{m_a}\right)'$. The first step to obtain an exact test for Δ_b, involves transforming the vector $\bar{\underset{\sim}{y}}$ into a vector whose expected value is the zero vector.

Consider the following tranformation:

$$\underset{\sim}{z}_M = C\bar{\underset{\sim}{y}}, \tag{3.25}$$

where $C = \bigoplus_{i=1}^{a} T_i^*$ and

$$T_i^* = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \ldots\ldots\ldots\ldots\ldots & & \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \ldots\ldots\ldots & \\ & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \dfrac{1}{\sqrt{m_i(m_i-1)}} & \dfrac{1}{\sqrt{m_i(m_i-1)}} & \ldots\ldots\ldots\ldots\ldots\ldots & & \dfrac{-(m_i-1)}{\sqrt{m_i(m_i-1)}} \end{bmatrix} \tag{3.26}$$

The matrix T_i^* for i=1,2,...,a has m_i-1 rows and m_i columns and is of rank m_i-1. Hence, the matrix C has $r_1 = \sum_{i=1}^{a} (m_i-1)$ rows and $m_\bullet$ columns. Each row of C defines a contrast in the elements of $\bar{\underset{\sim}{y}}$ and the rows are orthogonal to each other (that is, rank(C) = r_1). Therefore, we have the following results:

$$\begin{aligned} E(\underset{\sim}{z}_M) &= CE(\bar{\underset{\sim}{y}}) \\ &= \mu C\underset{\sim}{1}_{m_\bullet} + CB_1\underset{\sim}{\alpha} = \underset{\sim}{0}, \end{aligned} \tag{3.27}$$

where $\underset{\sim}{0}$ is a vector of zeroes of order $(r_1 \times 1)$. In addition,

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{z}_M) &= C[\mathrm{Var}(\bar{\underset{\sim}{y}})]C' \\ &= C[I_{m_\bullet}\sigma_b^2 + K\sigma_e^2]C' \\ &= CC'\sigma_b^2 + CKC'\sigma_e^2 \\ &= I_{r_1}\sigma_b^2 + K_1\sigma_e^2, \end{aligned} \tag{3.28}$$

where $K_1 = CKC'$ (recall from (3.4), $K = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{m_i} n_{j(i)}^{-1}$); $CC' = I_{r_1}$ follows from (3.26). Now consider,

$$\begin{aligned} Q_b^{(M)} &= \underset{\sim}{z}_M' \left(I_{r_1}\sigma_b^2 + K_1\sigma_e^2\right)^{-1} \underset{\sim}{z}_M \\ &= \underset{\sim}{z}_M' \left(I_{r_1}\Delta_b + K_1\right)^{-1} \underset{\sim}{z}_M / \sigma_e^2 \ . \end{aligned} \tag{3.29}$$

From (3.27) and (3.28), $Q_b^{(M)}$ has a central chi-squared distribution. In addition, $Q = \sum_{ijk} (y_{ijk} - \bar{y}_{j(i)})^2$ is independently distributed of $Q_b^{(M)}$ since $Q_b^{(M)}$ depends on the observations through the column vector of means, $\bar{\underset{\sim}{y}} = \{\bar{y}_{ij.}\}$ and $\bar{y}_{ij.}$ is

independently distributed from $\sum_{k=1}^{n_{ij}} (y_{ijk} - \bar{y}_{ij.})^2 \quad ij$ (see

Lindgren (1976) p.333, Theorem 3). Furthermore, it is known that $Q/\sigma_e^2 \sim \chi^2(0)$ with $n_{\bullet} - m_{\bullet}$ degrees of freedom. Hence,

$$F^M(\Delta_b) = \frac{\underset{\sim}{z}_M' \left(I_{r_1}\Delta_b + K_1\right)^{-1} \underset{\sim}{z}_M / r_1}{\sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2 / (n_{..} - m_{.})}, \qquad (3.30)$$

is distributed as a central F with numerator and demoninator degrees of freedom $r_1 = m_{.} - a$ and $n_{..} - m_{.}$, respectively.

We can use $F^M(\Delta_o)$ for testing

$$H_o\colon \Delta_b < \Delta_o \text{ against } H_a\colon \Delta_b > \Delta_o \qquad (3.31)$$

where Δ_o is some known quantity. Note that $F^M(\Delta_b)$ is a decreasing function of Δ_b, hence, we reject the null hypothesis for large values of $F^M(\Delta_o)$.

3.2.3 Derivation of an Exact Test for the Fixed Effects

In this section we derive an exact test for testing any hypothesis that can be expressed in terms of estimable linear functions of the fixed effects. Recall from Searle (1971, p. 180) that a linear function of the parameters is defined as estimable if it is a linear function of the expected value of the vector of observations $\underset{\sim}{y}$.

Consider again $Q = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2$, which can be expressed as

$$Q = \underset{\sim}{y}' W \underset{\sim}{y}, \qquad (3.32)$$

where $W = \left\{I_{n..} - \bigoplus_{i,j} \left(J_{n_{j(i)}} / n_{j(i)}\right)\right\}$. The matrix W is obviously symmetric and has rank equal to $n_{..} - m_.$. Hence, there exists an orthogonal matrix H such that

$$H'WH = \Lambda_2 \text{ or } W = H\Lambda_2 H', \qquad (3.33)$$

where Λ_2 is a diagonal matrix with diagonal elements equal to the eigenvalues of W. Since W is idempotent it has eigenvalues equal to zero or unity (Graybill, 1983 p.419). Thus, Λ_2 is a diagonal matrix with $n_{..} - m_.$ diagonal elements equal to unity and the remaining $m_.$ diagonal elements equal to zero. Assuming that $n_{..} > 2m_.$, we can partition H and Λ_2 as follows:

$$\begin{aligned} H &= \left(H_1 : H_2 : H_3\right) \\ \Lambda_2 &= \text{Diag}\left(I_{m_.}, I_{n_{..}-2m_.}, \underset{\sim}{0}\right), \end{aligned} \qquad (3.34)$$

where H_1, H_2, and H_3 are matrices of orders $n_{..} \times m_{.}$, $n_{..} \times (n_{..} - 2m_{.})$, and $n_{..} \times m_{.}$, respectively; $\underset{\sim}{0}$ is an $m_{.} \times m_{.}$ matrix of zeroes. Note that

$$\begin{aligned} H_i' H_i &= I, \qquad \text{for } i = 1,2,3 \\ H_i' H_j &= \underset{\sim}{0}, \qquad \text{for } i \neq j \,. \end{aligned} \tag{3.35}$$

From (3.34) - (3.35) we have

$$\begin{aligned} Q &= \underset{\sim}{y}' W \underset{\sim}{y} \\ &= \underset{\sim}{y}' H_1 H_1' \underset{\sim}{y} + \underset{\sim}{y}' H_2 H_2' \underset{\sim}{y} \\ &= Q_1 + Q_2 , \end{aligned} \tag{3.36}$$

where $Q_1 = \underset{\sim}{y}' H_1 H_1' \underset{\sim}{y}$; $Q_2 = \underset{\sim}{y}' H_2 H_2' \underset{\sim}{y}$.

Now consider

$$\underset{\sim}{w} = \bar{\underset{\sim}{y}} + (\lambda_{\max} I_{m_{.}} - K)^{1/2} H_1' \underset{\sim}{y}, \tag{3.37}$$

where $\lambda_{\max}$ is the largest eigenvalue of K, that is, $\lambda_{\max}$ is the inverse of the smallest $n_{j(i)}$. The column vector $\underset{\sim}{y}$ of observations can be expressed as

$$\underset{\sim}{y} = \mu \underset{\sim}{1}_{n_{..}} + X_1 \underset{\sim}{\alpha} + X_2 \underset{\sim}{\beta} + \underset{\sim}{e}, \tag{3.38}$$

where X_1 and X_2 are matrices of zeroes and ones of orders $n_{..} \times a$ and $n_{..} \times m_{.}$, respectively.

Lemma 3.2

The matrix $(\lambda_{\max} I_{m_\bullet} - K)^{1/2}$ is positive semidefinite.

Proof: Recall that $K = \mathrm{Diag}\left(n_{1(1)}^{-1}, n_{2(1)}^{-1}, \ldots, n_{m_a(a)}^{-1}\right)$ and $\lambda_{\max} = \left\{\min_{ij} n_{j(i)}\right\}^{-1}$, hence,

$$\lambda_{\max} - n_{j(i)}^{-1} \geq 0 \qquad \forall_{i,j} \, .$$

Consequently, $(\lambda_{\max} I_{m_\bullet} - K)$, which is a diagonal matrix, is positive semidefinite. Hence, $(\lambda_{\max} I_{m_\bullet} - K)^{1/2}$ is a positive semidefinite diagonal matrix. □

It follows from lemma 3.2, that the matrix $(\lambda_{\max} I_{m_\bullet} - K)^{1/2}$ is a well-defined matrix.

Lemma 3.3

$$\mathrm{Cov}(\bar{\underset{\sim}{y}}, \underset{\sim}{y}' H_1) = 0 \, .$$

Proof: We have that $\bar{\underset{\sim}{y}} = G\underset{\sim}{y}$, where

$$G = \mathrm{Diag}\left(n_{1(1)}^{-1} \underset{\sim}{1}'_{n_{1(1)}}, \; n_{2(1)}^{-1} \underset{\sim}{1}'_{n_{2(1)}}, \; \ldots, n_{m_a(a)}^{-1} \underset{\sim}{1}'_{m_a(a)}\right) .$$

Let Σ be the variance-covariance matrix of $\underset{\sim}{y}$. From Lindgren (1976, p.333, Thm 3), $\bar{\underset{\sim}{y}}$ and the residual sum of squares, Q, are statistically independent, hence from Searle (1971 p.59 Thm 3), we have $G\Sigma W = \underset{\sim}{0}$ (recall from (3.32) that $Q = \underset{\sim}{y}' W \underset{\sim}{y}$). But,

$$G\Sigma W = \underset{\sim}{0} \iff G\Sigma\left(H_1 H_1' + H_2 H_2'\right) = \underset{\sim}{0} \qquad \text{(see (3.41))}$$

$$\Rightarrow G\Sigma(H_1H_1' + H_2H_2')H_1 = \underset{\sim}{0}$$

$$\Leftrightarrow G\Sigma H_1 = \underset{\sim}{0}. \qquad \text{(see (3.40))}$$

Then, $\text{Cov}(\bar{\underset{\sim}{y}}, \underset{\sim}{y}'H_1) = \text{Cov}(G\underset{\sim}{y}, \underset{\sim}{y}'H_1) = G\Sigma H_1 = 0.$ □

Lemma 3.4

(i) $H_1'\Sigma H_1 = \sigma_e^2 I_m.$

(ii) $W\underset{\sim}{1}_{n_{\cdot\cdot}} = \underset{\sim}{0}$; $WX_1 = \underset{\sim}{0}$.

Proof: (i) The matrix Σ, the variance-covariance matrix of $\underset{\sim}{y}$, can be expressed as

$$\Sigma = X_2X_2'\sigma_b^2 + I_{n_{\cdot\cdot}}\sigma_e^2, \qquad (3.39)$$

where $X_2 = \bigoplus_{i,j} \underset{\sim}{1}_{n_{j(i)}}$. Consider the matrix product WX_2:

$$WX_2 = [I_{n_{\cdot\cdot}} - \bigoplus_{i,j}(n_{j(i)}^{-1}J_{n_{j(i)}})] \bigoplus_{i,j}\underset{\sim}{1}_{n_{j(i)}}$$

$$= \bigoplus_{i,j}\underset{\sim}{1}_{n_{j(i)}} - \bigoplus_{i,j}(n_{j(i)}^{-1}J_{n_{j(i)}})[\bigoplus_{i,j}\underset{\sim}{1}_{n_{j(i)}}]$$

$$= \bigoplus_{i,j}\underset{\sim}{1}_{n_{j(i)}} - \bigoplus_{i,j}[n_{j(i)}^{-1}J_{n_{j(i)}}\underset{\sim}{1}_{n_{j(i)}}]$$

$$= \bigoplus_{i,j}\underset{\sim}{1}_{n_{j(i)}} - \bigoplus_{i,j}\underset{\sim}{1}_{n_{j(i)}} = \underset{\sim}{0}_{n_{\cdot\cdot}\times m}.$$

Hence,

$$(H_1H_1' + H_2H_2')X_2 = \underset{\sim}{0}$$

$$\Rightarrow H_1'(H_1H_1' + H_2H_2')X_2 = \underset{\sim}{0}$$

$$\Leftrightarrow H_1'X_2 = \underset{\sim}{0}, \text{ since } H_1'H_2 = \underset{\sim}{0} \text{ from (3.35).}$$

From (3.39) we have

$$H_1'\Sigma H_1 = H_1'(X_2X_2'\sigma_b^2 + I_{m.}\sigma_e^2)H_1$$

$$= H_1'X_2X_2'H_1\sigma_b^2 + H_1'H_1\sigma_e^2$$

$$= H_1'H_1\sigma_e^2 \qquad (\text{since } H_1'X_2 = \underset{\sim}{0})$$

$$= \sigma_e^2 I_{m.} \quad .$$

(ii) The matrix $X_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}}$ (where $n_{i.} = \sum_{j=1}^{m_i} n_{ij}$). We have

$$WX_1 = [I_{n..} - \bigoplus_{i,j} (n_{j(i)}^{-1} J_{n_{j(i)}})] \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}}$$

$$= \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} - [\bigoplus_{i,j} (n_{j(i)}^{-1} J_{n_{j(i)}})][\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}}].$$

But, $\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}}$ can be expressed as

$$\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} = \bigoplus_{i=1}^{a} [(\bigoplus_{j=1}^{m_i} \underset{\sim}{1}_{n_{j(i)}})\underset{\sim}{1}_{m_i}];$$

hence,

$$WX_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} - [\bigoplus_{i,j} (n_{j(i)}^{-1} J_{n_{j(i)}})][\bigoplus_{i=1}^{a} (\bigoplus_{j=1}^{m_i} \underset{\sim}{1}_{n_{j(i)}})\underset{\sim}{1}_{m_i}]$$

$$= \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} - \bigoplus_{i=1}^{a} \left[\left(\bigoplus_{j=1}^{m_i} n_{j(i)}^{-1} J_{n_{j(i)}} \underset{\sim}{1}_{n_{j(i)}}\right) \underset{\sim}{1}_{m_i}\right]$$

$$= \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} - \bigoplus_{i=1}^{a} \left[\left(\bigoplus_{j=1}^{m_i} \underset{\sim}{1}_{n_{j(i)}}\right) \underset{\sim}{1}_{m_i}\right]$$

$$= \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} - \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} = \underset{\sim}{0}_{n_{..} \times a}.$$

Furthermore,

$$W \underset{\sim}{1}_{n_{..}} = \left[I_{n_{..}} - \bigoplus_{i,j} \left(n_{j(i)}^{-1} J_{n_{j(i)}}\right)\right]\left[\bigoplus_{i,j} \underset{\sim}{1}_{n_{j(i)}}\right] \underset{\sim}{1}_{m_{.}}$$

$$= \underset{\sim}{1}_{n_{..}} - \left[\bigoplus_{i,j} \underset{\sim}{1}_{n_{j(i)}}\right] \underset{\sim}{1}_{m_{.}} = \underset{\sim}{1}_{n_{..}} - \underset{\sim}{1}_{n_{..}} = \underset{\sim}{0}_{n_{..} \times 1} \quad \square$$

Using the results of lemmas 3.3 and 3.4, we have

$$E(\underset{\sim}{W}) = E(\bar{\underset{\sim}{y}}) + (\lambda_{\max} I_{m_{.}} - K)^{1/2} H_1' E(\underset{\sim}{y}) \tag{3.40}$$

$$= \mu \underset{\sim}{1}_{m_{.}} + B_1 \underset{\sim}{\alpha} + (\lambda_{\max} I_{n_{..}} - K)^{1/2} H_1' (\mu \underset{\sim}{1}_{n_{..}} + X_1 \underset{\sim}{\alpha})$$

$$= \mu \underset{\sim}{1}_{m_{.}} + B_1 \underset{\sim}{\alpha} .$$

Note that $H_1'(\mu \underset{\sim}{1}_{n_{..}} + X_1 \underset{\sim}{\alpha}) = \underset{\sim}{0}$ by lemma 3.4(ii) and (3.35), since $W = H_1 H_1' + H_2 H_2'$. In addition, the variance-covariance matrix of $\underset{\sim}{w}$ can be expressed as

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{w}) &= \mathrm{Var}(\bar{\underset{\sim}{y}}) + (\lambda_{\max} I_{m_{.}} - K)^{1/2} H_1' \mathrm{Var}(\underset{\sim}{y}) H_1 (\lambda_{\max} I_{m_{.}} - K)^{1/2} \\ &= \mathrm{Var}(\bar{\underset{\sim}{y}}) + (\lambda_{\max} I_{m_{.}} - K)^{1/2} H_1' \Sigma H_1 (\lambda_{\max} I_{m_{.}} - K)^{1/2} \end{aligned}$$

$$= \mathrm{Var}(\underset{\sim}{\bar{y}})+(\lambda_{\max} I_{m.} -K)^{1/2}(\lambda_{\max} I_{m.} -K)^{1/2}\sigma_e^2 \quad \text{(see lemma 3.4(i))}$$

$$= I_{m.}\sigma_b^2 + K\sigma_e^2 + (\lambda_{\max} I_{m.} -K)\sigma_e^2$$

$$= I_{m.}\left(\sigma_b^2 + \lambda_{\max}\sigma_e^2\right) . \tag{3.41}$$

Consequently, $\underset{\sim}{w}$ can be expressed as

$$\underset{\sim}{w} = \mu\underset{\sim}{1}_{m.} + B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + \underset{\sim}{\bar{e}} + \left(\lambda_{\max} I_{m.} -K\right)^{1/2} H_1'\underset{\sim}{y}, \tag{3.42}$$

but, $H_1'\underset{\sim}{y} = H_1'(\mu\underset{\sim}{1}_{n..} + X_1\underset{\sim}{\alpha} + X_2\underset{\sim}{\beta} + \underset{\sim}{e})$; hence, from lemma 3.4, $H_1'\underset{\sim}{y} = H_1'\underset{\sim}{e}$ and we have

$$\underset{\sim}{w} = \mu\underset{\sim}{1}_{m.} + B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + \underset{\sim}{\bar{e}} + (\lambda_{\max} I_{m.} -K)^{1/2} H_1'\underset{\sim}{e} . \tag{4.43}$$

Let X, $\underset{\sim}{\rho}$ and $\underset{\sim}{e}^*$ be defined as

$$\begin{aligned} X &= [\underset{\sim}{1}_{m.} : B_1], \\ \underset{\sim}{\rho}^* &= [\mu : \underset{\sim}{\alpha}']', \\ \text{and} \qquad \underset{\sim}{e}^* &= B_2\underset{\sim}{\beta} + \underset{\sim}{\bar{e}} + (\lambda_{\max} I_{m.} -K)^{1/2} H_1'\underset{\sim}{e}. \end{aligned} \tag{3.44}$$

Using (3.44) we have

$$\underset{\sim}{w} = X\underset{\sim}{\rho}^* + \underset{\sim}{e}^* . \tag{3.45}$$

Note that from (3.41) and model definitions,

$\underset{\sim}{e}^{*} \sim N_{m_{\bullet}}\{\underset{\sim}{0}, I_{m_{\bullet}}(\sigma_b^2 + \lambda_{\max}\sigma_e^2)\}$. Consider the normal equations stemming from the generalized least squares of (3.45), that is,

$$X'[\mathrm{Var}(\underset{\sim}{w})]^{-1}X\underset{\sim}{\beta}^{o} = X'[\mathrm{Var}(\underset{\sim}{w})]^{-1}\underset{\sim}{w}$$

$$\Longleftrightarrow \quad X'X\underset{\sim}{\beta}^{o} = X'\underset{\sim}{w} \qquad \text{(see (3.41) above)},$$

with a solution

$$\underset{\sim}{\beta}^{o} = (X'X)^{-}X'\underset{\sim}{w} \tag{3.46}$$

where $(X'X)^{-}$ represents a generalized inverse of the matrix $X'X$. We are interested in testing

$$H_o: \; R'\underset{\sim}{\beta} = \underset{\sim}{m}, \tag{3.47}$$

where R' has full row rank (let $\mathrm{rank}(R') = s$) and $R' = F'X$ for some matrix $F'_{s\times m_{\bullet}}$, that is, $R'\underset{\sim}{\beta}$ are estimable linear functions of $\underset{\sim}{\beta}$; $\underset{\sim}{m}$ is a vector, of order $s\times 1$ of specified constants. Now consider,

$$\underset{\sim}{\ell}_a^{(M)} = R'\underset{\sim}{\beta}^{o} - \underset{\sim}{m}\,. \tag{3.48}$$

Theorem 3.2.

$$Q_a^{(M)} = \underset{\sim}{\ell}_a^{(M)'} \left(R'G_X R\right)^{-1} \underset{\sim}{\ell}_a^{(M)} / \sigma^{*2}$$

$$\sim \chi^{2'}\left[s,\ (R'\underset{\sim}{\rho}-\underset{\sim}{m})'(R'G_X R)^{-1}(R'\underset{\sim}{\rho}-\underset{\sim}{m})/2\sigma^{*2}\right],$$

$$G_X = (X'X)^- \ ; \ \sigma^{*2} = \sigma_b^2 + \lambda_{\max}\sigma_e^2 \ .$$

Proof: $E\left(\underset{\sim}{\ell}_a^{(M)}\right) = R'E(\underset{\sim}{\rho}^{o}) - \underset{\sim}{m} = R'(X'X)^- X'X\underset{\sim}{\rho} - \underset{\sim}{m}$

$$= F'X(X'X)^- X'X\underset{\sim}{\rho} - \underset{\sim}{m} = F'X\,\underset{\sim}{\rho} - \underset{\sim}{m} = R'\underset{\sim}{\rho} - \underset{\sim}{m}$$

and $\operatorname{Var}\left(\underset{\sim}{\ell}_a^{(M)}\right) = R'\operatorname{Var}(\underset{\sim}{\rho}^{o})R$

$$= F'X(X'X)^- X'\left[\operatorname{Var}(\underset{\sim}{w})\right]X(X'X)^- X'F$$

$$= F'X(X'X)^- X'X(X'X)^- X'F\left(\sigma_b^2 + \lambda_{\max}\sigma_e^2\right)$$

$$= R'(X'X)^- R\left(\sigma_b^2 + \lambda_{\max}\sigma_e^2\right) \ .$$

Hence,

$$\underset{\sim}{\ell}_a^{(M)} \sim N_s\left[R'\underset{\sim}{\rho} - \underset{\sim}{m},\ R'G_X R\left(\sigma_b^2 + \lambda_{\max}\sigma_e^2\right)\right]$$

Therefore, from Searle (1971, p.57, Thm 2), the proof is complete. □

Now consider

$$SSE^{(M)} = \underset{\sim}{w}'\underset{\sim}{w} - \underset{\sim}{\rho}^{o'} X'\underset{\sim}{w} \tag{3.49}$$

$$= \underset{\sim}{w}'\underset{\sim}{w} - \underset{\sim}{w}'X(X'X)^{-}X'\underset{\sim}{w}$$

$$= \underset{\sim}{w}'\left(I_{m_{\bullet}} - X(X'X)^{-}X'\right)\underset{\sim}{w} \; .$$

Theorem 3.3

$$SSE^{(M)}/\sigma^{*2} \sim \chi^{2}_{m_{\bullet}-a} .$$

Proof: We know that $I_{m_{\bullet}} - X(X'X)^{-}X'$ is an idempotent matrix with

$$\text{rank}\left(I_{m_{\bullet}} - X(X'X)^{-}X'\right) = \text{trace}\left(I_{m_{\bullet}} - X(X'X)^{-}X'\right)$$

$$= \text{rank}\left(I_{m_{\bullet}}\right) - \text{rank}(X)$$

$$= m_{\bullet} - a \; .$$

Furthermore, $E(\underset{\sim}{w}) = X\underset{\sim}{\rho}$, hence,

$$[E(w)]'[I_{m_{\bullet}} - X(X'X)X'][E(\underset{\sim}{w})]$$

$$= \underset{\sim}{\rho}'X'[I_{m_\bullet} - X(X'X)'X']X\underset{\sim}{\rho}$$

$$= \underset{\sim}{0}.$$

Hence, from Searle (1971, p.57, Theorem 2),

$$SSE^{(M)}/\sigma^{*2} \sim \chi^2_{m_\bullet - a}. \quad \square$$

In addition, we have the following theorem,

Theorem 3.4

$Q_a^{(M)}$ and $SSE^{(M)}$ are independently distributed.

Proof:

$$\sigma^{*2}Q_a^{(M)} = (R'\underset{\sim}{\rho}^o - \underset{\sim}{m})'(R'G_XR)^{-1}(R'\underset{\sim}{\rho}^o - \underset{\sim}{m})$$

$$= (R'(X'X)^-X'\underset{\sim}{w} - \underset{\sim}{m})'(R'G_XR)^{-1}(R'(X'X)^-X'\underset{\sim}{w} - \underset{\sim}{m}) \; .$$

We can express $\underset{\sim}{m}$ as

$$\underset{\sim}{m} = R'R(R'R)^{-1}\underset{\sim}{m} = F'XR(R'R)^{-1}\underset{\sim}{m} = F'XG_XX'XR(R'R)^{-1}\underset{\sim}{m}$$

$$= R'G_XX'XR(R'R)^{-1}\underset{\sim}{m} \; .$$

Hence,

$$\sigma^{*2} Q_a^{(M)} = (\underset{\sim}{w} - XR(R'R)^{-1}\underset{\sim}{m})'[XG_X R(R'G_X R)^{-1} R'G_X X'](\underset{\sim}{w} - XR(R'R)^{-1}\underset{\sim}{m}) \ .$$

Now,

$$SSE^{(M)} = \underset{\sim}{w}'[I_{m_\bullet} - X(X'X)^{-}X']\underset{\sim}{w}$$

$$= (\underset{\sim}{w} - XR(R'R)^{-1}\underset{\sim}{m})'(I_{m_\bullet} - XG_X X')(\underset{\sim}{w} - XR(R'R)^{-1}\underset{\sim}{m})$$

since $X'(I_{m_\bullet} - XG_X X') = \underset{\sim}{0}$. We have

$$XG_X R(R'G_X R)^{-1} R'G_X X'[Var(\underset{\sim}{w} - XR(R'R)^{-1}\underset{\sim}{m})](I_{m_\bullet} - XG_X X')$$

$$= XG_X R(R'G_X R)^{-1} R'G_X X'(I_{m_\bullet} - XG_X X')(\sigma_b^2 + \lambda_{max}\sigma_e^2)$$

$$= \underset{\sim}{0} \ .$$

It follows from Searle (1971, p.59, Theorem 4) that $Q_a^{(M)}$ and $SSE^{(M)}$ are independently distributed. □

Consequently, it follows from theorems 3.2, 3.3, and 3.4 that

$$F(H) = \frac{\underset{\sim}{\ell}_a^{(M)'}(R'G_X R)^{-1}\underset{\sim}{\ell}_a^{(M)}/s}{SSE^{(M)}/(m_\bullet - a)} \ . \qquad (3.50)$$

$$\sim F'[s,\ (m_\bullet - a),\ (R'\underset{\sim}{\rho} - \underset{\sim}{m})'(R'G_X R)^{-1}(R'\underset{\sim}{\rho} - \underset{\sim}{m})/2\sigma^{*2}]$$

Under the null hypothesis, H_o: $R'\underset{\sim}{\rho} = \underset{\sim}{m}$, the noncentrality parameter is zero and $F(H) \sim F_{s,(m_{\bullet}-a)}$, that is, F(H) has a central F distribution. Hence, F(H) provides an exact test of the hypothesis: $R'\underset{\sim}{\rho} = \underset{\sim}{m}$. In particular, we can test the hypothesis H_o: $\alpha_1 = \alpha_2 = \cdots = \alpha_a$.

Consider the following example where a manufacturer is interested in studying the burning rate of powder from three production processes (the fixed effects). Three batches (the random effects) of powder are randomly selected from each process and three determinations of burning rate (in minutes) are made on each batch. The results are shown below, in Table 3.1, where some of the data have been deleted to illustrate the analysis of an unbalanced design.

Table 3.1 Example of an Unbalanced 2-Fold Nested Mixed Model (Source: Montgomery (1976 problem 11-2, p. 281))

Process	Batch	Burning Rate
1	1	25, 30
	2	19
	3	15, 17, 14
2	1	19, 17, 14
	2	23, 24, 21
	3	35, 27, 25
3	1	20
	2	25, 33

Using the program in Appendix E, we test the significance of the fixed nesting effect and the variance component associated with the random nested effect.

Results:

For testing H_o: $\Delta_b = 0$ vs H_a: $\Delta_b \neq 0$,

where $\Delta_b = \sigma_b^2/\sigma_e^2$.

test statistic = 7.53 with numerator df = 5

denominator df = 10

p-value = .004

For testing the $H_o = \alpha_1 = \alpha_2 = \alpha_3$ (i.e., the fixed effects)

test statistic = .288 with numerator df = 2

denominator df = 5

p-value = .762

3.3 Reduction of the Exact Tests when the Design is Balanced.

In this section we investigate what happens to the exact tests derived in the last section when the design considered is a balanced design. The conventional tests applied when the design is balanced have been shown to have optimal properties; hence, a desirable property for the derived exact tests, is that they reduce to these conventional tests when the design is balanced. If the exact tests reduce to optimal tests under a balanced design future research on optimal tests for unbalanced designs may commence with these newly derived exact tests.

Consider the 2-fold nested mixed model in (3.21), but with $i=1,2,\ldots,a$; $j=1,2,\ldots,m$; $k=1,2,\ldots,n$, that is, the design is now balanced.

3.3.1 Reduction of the Exact Test for the Fixed Effects.

Recall $\underset{\sim}{w}$ in (3.37)

$$\underset{\sim}{w} = \bar{\underset{\sim}{y}} + (\lambda_{\max} I_{m_{\cdot}} - K)^{1/2} H_1' \underset{\sim}{y} \ .$$

Under the balanced situation we have $n_{j(i)} = n_{ij}$, hence, the matrix K reduces to

$$K = \underset{i,j}{\oplus} n_{j(i)}^{-1} = \underset{i,j}{\oplus} n^{-1} = n^{-1} I_{am}, \tag{3.51}$$

where $am = \sum_{i=1}^{a} m$. Therefore, λ_{max}, the largest eigenvalue of K, becomes $1/n$. Consequently, under a balanced design,

$$\underset{\sim}{w} = \bar{\underset{\sim}{y}} = \mu \underset{\sim}{1}_{am} + B_1 \underset{\sim}{\alpha} + I_{am} \underset{\sim}{\beta} + \bar{\underset{\sim}{e}}, \tag{3.52}$$

where $B_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_m = I_a \times \underset{\sim}{1}_m$. Proceeding as in (3.44) we now have

$$\begin{aligned} X &= [\underset{\sim}{1}_{am} : B_1], \\ \underset{\sim}{\rho} &= [\mu : \underset{\sim}{\alpha}']', \qquad\qquad (3.53) \\ \text{and} \qquad \underset{\sim}{e}^* &= I_{am} \underset{\sim}{\beta} + \bar{\underset{\sim}{e}} . \end{aligned}$$

Hence, alternatively, (3.52) can be written as

$$\underset{\sim}{w} = X \underset{\sim}{\rho} + \underset{\sim}{e}^* . \tag{3.54}$$

A solution to the normal equations stemming from model (3.54) is

$$\underset{\sim}{\rho}^o = (X'X)^- X' \bar{\underset{\sim}{y}} . \tag{3.55}$$

Lemma 3.5

$$\underset{\sim}{\rho}^o = \begin{bmatrix} 0 \\ \frac{1}{mn} B_1' (I_{am} \times \underset{\sim}{1}_n') \underset{\sim}{y} \end{bmatrix} ,$$

that is, $\mu^{o} = 0$ and $\underset{\sim}{\beta}^{o} = \frac{1}{mn} B_1'(I_{am} \times \underset{\sim}{1}_n')\underset{\sim}{y}$ is a solution to the normal equations of model (3.54).

Proof: The matrix $X'X$ can be expressed as

$$X'X = \begin{bmatrix} \underset{\sim}{1}_{am}' \\ \\ B_1' \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{am} & \vdots & B_1 \end{bmatrix} = \begin{bmatrix} am & m\underset{\sim}{1}_a' \\ \\ m\underset{\sim}{1}_a & mI_a \end{bmatrix} .$$

Now, rank($X'X$) = rank(X) = a, and applying the algorithm described in Searle (1982, p.218), a generalized inverse of $X'X$ is given by

$$(X'X)^{-} = \begin{bmatrix} 0 & \underset{\sim}{0}_a' \\ \\ \underset{\sim}{0}_a & \frac{1}{m} I_a \end{bmatrix} . \qquad (3.56)$$

Hence,

$$\underset{\sim}{\rho}^{o} = (X'X)^{-}X'\bar{\underset{\sim}{y}}$$

$$= \begin{bmatrix} 0 & \underset{\sim}{0}_a' \\ \\ \underset{\sim}{0}_a & \frac{1}{m} I_a \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{am}' \\ \\ B_1' \end{bmatrix} \bar{\underset{\sim}{y}}$$

$$= \begin{bmatrix} \underset{\sim}{0}'_{am} \\ \frac{1}{m} B_1' \end{bmatrix} \bar{\underset{\sim}{y}} .$$

But, $\bar{\underset{\sim}{y}} = n^{-1}(I_{am} \times \underset{\sim}{1}_n')\underset{\sim}{y}$; hence, we have

$$\underset{\sim}{\mu}^o = 0 \quad \text{and } \underset{\sim}{\alpha}^o = \frac{1}{mn} B_1'(I_{am} \times \underset{\sim}{1}_n')\underset{\sim}{y}. \qquad \square$$

Now, recall from (3.50) that

$$F(H) = \frac{(R'\underset{\sim}{\rho}^o - \underset{\sim}{m})'(R'G_X R)^{-1}(R'\underset{\sim}{\rho}^o - \underset{\sim}{m})/s}{\underset{\sim}{w}'(I_{m.} - X(X'X)^- X')\underset{\sim}{w}/(m_. - a)} ,$$

is the test statistic derived to test the H_o: $R'\underset{\sim}{\rho} = \underset{\sim}{m}$, for an unbalanced design. From lemma 3.5 we have seen a solution to the vector $\underset{\sim}{\rho}^o$ when we have a balanced situation. What happens to the denominator of F(H) when we have a balanced design?

Lemma 3.6

Under a balanced design $\underset{\sim}{w}'(I_{m.} - X(X'X)^- X')\underset{\sim}{w}$ becomes

$$\frac{1}{n} \underset{\sim}{y}'[n^{-1}(I_{am} \times J_n) - (mn)^{-1}(I_a \times J_{mn})]\underset{\sim}{y} .$$

Proof: From (3.52), we know that under a balanced design $\underset{\sim}{w} = \bar{\underset{\sim}{y}}$ and $m_. = am$. Hence,

$$\underset{\sim}{w}\left(I_{m.} - X(X'X)^{-}X'\right)\underset{\sim}{w} = \bar{\underset{\sim}{y}}'\left(I_{am} - X(X'X)^{-}X'\right)\bar{\underset{\sim}{y}} \,.$$

From (3.53) and (3.56),

$$X(X'X)^{-}X' = \begin{bmatrix} \underset{\sim}{1}_{am} & : & B_1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{0}_{1\times am} \\ \frac{1}{m}B_1' \end{bmatrix}$$

$$= \frac{1}{m}B_1B_1' = \frac{1}{m}\left(I_a \times \underset{\sim}{1}_m\right)\left(I_a \times \underset{\sim}{1}'_m\right)$$

$$= \frac{1}{m}\left(I_a \times J_m\right).$$

Consequently,

$$\underset{\sim}{w}'\left(I_{m.} - X(X'X)^{-}X'\right)\underset{\sim}{w}$$

$$= \bar{\underset{\sim}{y}}'\left(I_{am} - \frac{1}{m}\left(I_a \times J_m\right)\right)\bar{\underset{\sim}{y}}$$

$$= \frac{1}{n^2}\underset{\sim}{y}'\left(I_{am} \times \underset{\sim}{1}_n\right)\left[I_{am} - \frac{1}{m}\left(I_a \times J_m\right)\right]\left(I_{am} \times \underset{\sim}{1}'_n\right)\underset{\sim}{y}$$

$$= \frac{1}{n}\underset{\sim}{y}'\left[n^{-1}\left(I_{am} \times J_n\right) - (mn)^{-1}\left(I_a \times J_{mn}\right)\right]\underset{\sim}{y} \,. \quad \square$$

Hence, from lemmas 3.5 and 3.6 and (3.56), we have with a balanced design,

$$F(H) = \frac{\left(R'\underset{\sim}{\rho}^{0} - \underset{\sim}{m}\right)'\left(R'G_XR\right)^{-1}\left(R'\underset{\sim}{\rho}^{0} - \underset{\sim}{m}\right)/s}{n^{-1}\underset{\sim}{y}'\left[n^{-1}\left(I_{am}\times J_n\right) - (mn)^{-1}\left(I_a \times J_{mn}\right)\right]\underset{\sim}{y}/(am-a)} \,. \qquad (3.57)$$

Consider the conventional F used when we have a balanced 2-fold nested mixed model and we are interested in testing estimable linear functions of $\underset{\sim}{\rho}$, i.e., $R'\underset{\sim}{\rho}$. From Searle (1971, p.190), the conventional F-test can be expressed as

$$F_1 = \frac{(R'\underset{\sim}{\rho}^{*^o} - \underset{\sim}{m})'(R'G_{X^*}R)^{-1}(R'\underset{\sim}{\rho}^{*^o} - \underset{\sim}{m})/s}{\underset{\sim}{y}'[\bigoplus_{i=1}^{a}\{\bigoplus_{j=1}^{m} n^{-1}J_n - \frac{1}{mn}J_{mn}\}]\underset{\sim}{y}\,/(am-a)} ,$$

where $G_{X^*} = (X^{*'}X^*)^-$ and $X^* = X \times \underset{\sim}{1}_{n_{\cdot\cdot}}$;

$$\underset{\sim}{\rho}^{*^o} = [\mu^{*^o} : \underset{\sim}{\alpha}^{*^o}]'$$

$(\rho^{*^o} = \rho^o)$.

Note that we are using the superscript '*' since we use the y_{ijk}'s (i.e. model (3.38)) rather than the $\bar{y}_{ij.}$'s (i.e. model (3.23)) in deriving the conventional test.

Theorem 3.5

$$F(H) = F_1 .$$

Proof: First, consider the denominator of F_1. We have

$$\underset{\sim}{y}'[\bigoplus_{i=1}^{a}\{\bigoplus_{j=1}^{m}(n^{-1}J_n) - \frac{1}{mn}J_{mn}\}]\underset{\sim}{y}$$

$$= \underset{\sim}{y}'[\bigoplus_{i=1}^{a}\{n^{-1}(I_m \times J_n) - \frac{1}{mn}J_{mn}\}]\underset{\sim}{y}$$

$$= \underset{\sim}{y}'[n^{-1}(I_{am} \times J_n) - \frac{1}{mn}(I_a \times J_{mn})]\underset{\sim}{y}.$$

Now,

$$X^{*'}X^{*} = \begin{bmatrix} \underset{\sim}{1}'_{n_{\bullet\bullet}} \\ I_a \times \underset{\sim}{1}'_{mn} \end{bmatrix} [\underset{\sim}{1}_{n_{\bullet\bullet}} : I_a \times \underset{\sim}{1}_{mn}]$$

$$= \begin{bmatrix} amn & mn\underset{\sim}{1}'_a \\ mn\underset{\sim}{1}_a & mnI_a \end{bmatrix}$$

(note that $n_{\bullet\bullet} = amn$ when the design is balanced). Again, applying the algorithm described in Searle (1982, p.218) concerning a generalized inverse,

$$(X^{*'}X^{*})^{-} = \begin{bmatrix} 0 & \underset{\sim}{0}'_a \\ \underset{\sim}{0}_a & \frac{1}{mn}I_a \end{bmatrix}.$$

$$= \frac{1}{n}(X'X)^{-} \qquad \text{(see (3.56))}.$$

Consequently,

$$\underset{\sim}{\beta}^{*^{o}} = (X^{*'}X^{*})^{-}X^{*'}\underset{\sim}{y}$$

$$= \begin{bmatrix} 0 & \underset{\sim}{0}'_a \\ \underset{\sim}{0}_a & \frac{1}{mn}I_a \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}'_N \\ I_a \times \underset{\sim}{1}'_{mn} \end{bmatrix} \underset{\sim}{y}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{mn}\left(I_a \times \underset{\sim}{1}'_{mn}\right) \end{bmatrix} \underset{\sim}{y} = \underset{\sim}{\beta}^{o} \qquad \text{(see lemma 3.5)},$$

and therefore,

$$F_1 = \frac{n(R'\underset{\sim}{\beta}^{o} - \underset{\sim}{m})'(R'(X'X)^{-}R)^{-1}(R'\underset{\sim}{\beta}^{o} - \underset{\sim}{m})/s}{\underset{\sim}{y}'\left[n^{-1}(I_{am} \times J_n) - \frac{1}{mn}(I_a \times J_{mn})\right]\underset{\sim}{y}/(am-a)}$$

$$= F(H) \ . \qquad \square$$

We have demonstrated that the exact test, derived in the last section for testing estimble linear functions of the fixed effects when the design is unbalanced, reduces to the conventional F-test when the design is balanced.

3.3.2 Reduction of the Exact Test Concerning $\Delta_b = \sigma_b^2/\sigma_e^2$

Without loss of generality, let $\Delta_o = 0$, and recall from (3.30) that for testing $H_o : \Delta_b = 0$ vs $H_a : \Delta_b > 0$ we derived the following test statistic:

$$F(\Delta_o=0) = \frac{\underset{\sim}{z}_M' K_1^{-1} \underset{\sim}{z}_M / r_1}{\sum\limits_{ijk} (Y_{ijk} - \bar{y}_{ij.})^2/(n_{..} - m_{.})} ,$$

where $\underset{\sim}{z}_M = C\bar{\underset{\sim}{y}}$, $K_1 = CKC'$, $r_1 = \sum\limits_{i=1}^{a} (m_i - 1)$, and $m. = \sum\limits_{i=1}^{a} m_i$ (recall, $C = \bigoplus\limits_{i=1}^{a} T_i$, where T_i is defined in (3.26)).

Consider the conventional F-test employed when the design is balanced:

$$F_2 = \frac{\{\underset{\sim}{y}' \, [(I_{am} \times J_n)/n - (I_a \times J_{mn})/mn] \underset{\sim}{y}\} / a(m-1)}{\sum\limits_{ijk} (y_{ijk} - \bar{y}_{ij.})^2 / am(n-1)} , \qquad (3.59)$$

(see Khuri (1982)).

Theorem 3.6

$F^M(0) = F(\Delta_o = 0) = F_2$ when we have a balanced design.

Proof: When the design is balanced we have

$$\underset{\sim}{z}_M' K_1^{-1} \underset{\sim}{z}_M/r_1 = \underset{\sim}{z}_M'(n^{-1}I_{r_1})^{-1} \underset{\sim}{z}_M/r_1$$

$$= n \underset{\sim}{z}_M' \underset{\sim}{z}_M/r_1 .$$

But, using the matrix T defined in (3.26),

$$n\underset{\sim}{z}_M'\underset{\sim}{z}_M = n\bar{\underset{\sim}{y}}'C'C\bar{\underset{\sim}{y}}$$

$$= n\bar{\underset{\sim}{y}}'[I_a \times T'_{(m-1)\times m}][I_a \times T_{(m-1)\times m}]\bar{\underset{\sim}{y}}$$

$$= n\bar{\underset{\sim}{y}}'(I_a \times T'T)\bar{\underset{\sim}{y}}$$

$$= n\bar{\underset{\sim}{y}}'[I_a \times (I_m - \frac{1}{m} J_m)]\bar{\underset{\sim}{y}} ,$$

$$= n\bar{\underset{\sim}{y}}'[I_{am} - \frac{1}{m}(I_a \times J_m)]\bar{\underset{\sim}{y}} .$$

But, $\bar{\underset{\sim}{y}} = n^{-1}(I_{am} \times \underset{\sim}{1}_n')\underset{\sim}{y}$; hence,

$$n\underset{\sim}{z}_M'\underset{\sim}{z}_M = n^{-1}\underset{\sim}{y}'[(I_{am} \times \underset{\sim}{1}_n)(I_{am} \times \underset{\sim}{1}_n')$$

$$- \frac{1}{m} (I_{am} \times \underset{\sim}{1}_n)(I_a \times J_m)(I_{am} \times \underset{\sim}{1}_n')]\underset{\sim}{y}$$

$$= n^{-1} \underset{\sim}{y}'[(I_{am} \times J_n) - \frac{1}{m} (I_a \times J_m \times J_n)]\underset{\sim}{y}$$

$$= n^{-1} \underset{\sim}{y}'[(I_{am} \times J_n) - \frac{1}{m} (I_a \times J_{mn})]\underset{\sim}{y}$$

$$= \underset{\sim}{y}'[\frac{1}{n} (I_{am} \times J_n) - \frac{1}{mn} (I_a \times J_{mn})]\underset{\sim}{y} \; .$$

Furthermore,

$$r_1 = \sum_{i=1}^{a} (m_i - 1) = a(m-1);$$

$$m_{\bullet} = \sum_{i=1}^{a} m_i = am \; .$$

Consequently,

$$F^M(0) = \frac{\{\underset{\sim}{y}' \; [\frac{1}{n} (I_{am} \times J_n) - \frac{1}{mn} (I_a \times J_{mn})]\underset{\sim}{y}\} / a(m-1)}{\sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2 \; / \; (n_{\bullet\bullet} - am)}$$

$$= F_2 \; . \qquad \square$$

3.4 An Unbalanced Mixed Two-Way Crossed Classification Model with Interaction

3.4.1 Introduction

Consider the following model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} ;$$

$$i=1,2,\ldots,r;\ j=1,2,\ldots,s;\ k=1,2,\ldots,n_{ij}, \tag{3.60}$$

where μ is an unknown constant parameter; α_i are the unknown fixed parameters; β_j, $(\alpha\beta)_{ij}$, and e_{ijk} are independently normally distributed random variables with zero means and variances σ^2_β, $\sigma^2_{\alpha\beta}$, and σ^2_e, respectively.

Define $\bar{y}_{ij} = \sum_{k=1}^{n_{ij}} y_{ijk}/n_{ij}$. Then,

$$\bar{y}_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \bar{e}_{ij.} \quad ij, \tag{3.61}$$

where $\bar{e}_{ij.} = \sum_{k=1}^{n_{ij}} e_{ijk}/n_{ij}$. In matrix notation (3.61) can be expressed as

$$\bar{\underset{\sim}{y}} = \mu \underset{\sim}{1}_{rs} + B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + I_{rs}(\alpha\beta) + \bar{\underset{\sim}{e}} , \tag{3.62}$$

where $B_1 = I_r \times \underset{\sim}{1}_s$ and $B_2 = \underset{\sim}{1}_r \times I_s$; $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, and $\alpha\beta$ are column vectors whose elements are α_i for $i=1,2,\ldots,r$, β_j for

$j=1,2,\ldots,s$, and $(\alpha\beta)_{ij}$ for $\forall_{ij}$, respectively;

$\bar{\underset{\sim}{e}} = (\bar{e}_{11}, \bar{e}_{12},\ldots,\bar{e}_{rs})'$.

3.4.2 Derivation of an Exact Test for the Variance Components Associated with the Random Effects

The expectation of the vector $\bar{\underset{\sim}{y}}$ in (3.62) is

$$E(\bar{\underset{\sim}{y}}) = \mu \underset{\sim}{1}_{rs} + B_1 \underset{\sim}{\alpha}, \tag{3.63}$$

with a variance-covariance matrix

$$\text{Var}(\bar{\underset{\sim}{y}}) = A_2\sigma^2_\beta + I\sigma^2_{\alpha\beta} + K\sigma^2_e, \tag{3.64}$$

where $A_2 = B_2B_2' = (\underset{\sim}{1}_r \times I_s)(\underset{\sim}{1}_r' \times I_s) = J_r \times I_s$;

$K = \text{Diag}\,(n^{-1}_{11}, n^{-1}_{12}, \ldots, n^{-1}_{rs})$. Note from (3.63) that the expectation of the vector $\bar{\underset{\sim}{y}}$ involves the column vector $\underset{\sim}{\alpha}$. As in Section 3.2.2, we need to obtain a transformation that removes these nuisance parameters in order to obtain an exact test for the variance components associated with the random effects.

Consider the following transformation,

$$\underset{\sim}{z}_t = C_t \bar{\underset{\sim}{y}}, \tag{3.65}$$

where the subscript t indicates that we are considering a two-way crossed classification; $C_t = I_r \times T$, where T is a matrix now defined as

$$T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \quad 0 \ldots\ldots\ldots\ldots & 0 \\ \vdots & \vdots & & \vdots \\ 1/\sqrt{s(s-1)} & 1/\sqrt{s(s-1)} & \ldots\ldots\ldots\ldots\ldots\ldots & \dfrac{-(s-1)}{\sqrt{s(s-1)}} \end{bmatrix}. \qquad (3.66)$$

The matrix T has s-1 rows and s columns. Hence, the matrix C_t consists of r×(s-1) rows and rs columns. Each row of C_t defines a contrast on the elements composing that row, and the rows are orthogonal to each other. From (3.63), (3.64), (3.65) and (3.66) we have the following results:

$$E(\underset{\sim}{z}_t) = \underset{\sim}{0} \qquad (3.67)$$

$$\begin{aligned} Var(\underset{\sim}{z}_t) &= C_t A_2 C_t' \sigma_\beta^2 + C_t C_t' \sigma_{\alpha\beta}^2 + C_t K C_t' \sigma_e^2 \\ &= C_t A_2 C_t' \sigma_\beta^2 + I_{r\times(s-1)} \sigma_{\alpha\beta}^2 + C_t K C_t' \sigma_e^2 . \qquad (3.68) \end{aligned}$$

Note that $C_t C_t' = I_{r\times(s-1)}$, since $TT' = I_{s-1}$, and $C_t C_t' = (I_r \times T)(I_r \times T') = I_r \times TT'$. Hence, $C_t C_t' = I_r \times I_{s-1} = I_{r\times(s-1)}$.

The matrix, $C_tA_2C_t'$, has eigenvalues r and 0 with multiplicity equal to s-1 and (r-1)(s-1), respectively (See Appendix C for proof.). In addition, $C_tA_2C_t'$ is symmetric; hence, there exists, an orthogonal matrix P of order (rs-r)×(rs-r) such that $PC_tA_2C_t'P'$ $= D_t$, where D_t is a diagonal matrix whose diagonal elements are the eigenvalues of $C_tA_2C_t'$.

Consider the transformation

$$\underset{\sim}{\mu}_t = P\underset{\sim}{z}_t \tag{3.68}$$

From (3.67)

$$E(\underset{\sim}{\mu}_t) = \underset{\sim}{0} \tag{3.69}$$

and

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{\mu}_t) &= PC_tA_2C_t'P'\sigma^2_\beta + I_{r(s-1)}\sigma^2_{\alpha\beta} + PC_tKC_t'P'\sigma^2_e \\ &= D_t\sigma^2_\beta + I_{r(s-1)}\sigma^2_{\alpha\beta} + L_t\sigma^2_e \\ &= \mathrm{Diag}\left[\left(r\sigma^2_\beta + \sigma^2_{\alpha\beta}\right)I_{s-1},\ \sigma^2_{\alpha\beta}I_{(r-1)(s-1)}\right] + L_t\sigma^2_e. \end{aligned}$$

Where $L_t = PC_tKC_t'P'$.

Now, model (3.60) can be expressed in matrix notation as

$$\underset{\sim}{y} = \mu \underset{\sim}{1}_{n_{\cdot\cdot}} + X_1\underset{\sim}{\alpha} + X_2\underset{\sim}{\beta} + X_3(\alpha\beta) + \underset{\sim}{e}, \tag{3.70}$$

where X_1, X_2, and X_3 are matrices of zeroes and ones of orders $n_{\cdot\cdot}\times r$, $n_{\cdot\cdot}\times s$, and $n_{\cdot\cdot}\times rs$, respectively. We have

$$\Sigma_t = \mathrm{Var}(\underset{\sim}{y}) = X_2X_2'\sigma_\beta^2 + X_3X_3'\sigma_{\alpha\beta}^2 + I_{n_{\cdot\cdot}}\sigma_e^2. \tag{3.71}$$

Consider

$$Q_t = \sum_{ijk} (y_{ijk} - \bar{y}_{ij\cdot})^2, \tag{3.72}$$

which can be expressed as

$$Q_t = \underset{\sim}{y}'W_t\underset{\sim}{y}, \tag{3.73}$$

where $W_t = I_{n_{\cdot\cdot}} - \bigoplus_{i,j} (J_{n_{ij}}/n_{ij})$. The matrix W_t is idempotent and symmetric with rank equal to $n_{\cdot\cdot}-rs$. Hence, there exists an orthogonal matrix H_t such that

$$W_t = H_t\Lambda_3H_t', \tag{3.74}$$

where Λ_3 is a diagonal matrix whose diagonal elements are the eigenvalues of W_t. Hence, Λ_3 has $n_{\cdot\cdot}-rs$ diagonal elements equal to unity and rs diagonal elements equal to zero. We can partition Λ_3 and H_t in a manner similar to Khuri (1986b) and obtain

$$\Lambda_3 = \text{Diag}\left(I_{r(s-1)}, I_{(n_{..}-2rs+r)}, \underset{\sim}{0}\right) \tag{3.75}$$

and

$$H_t = \left(H_{t1} : H_{t2} : H_{t3}\right),$$

where $\underset{\sim}{0}$ is a zero matrix of order rs×rs and H_{t1}, H_{t2}, and H_{t3} are matrices of orders $n_{..}\times r(s-1)$, $n_{..}\times(n_{..}-2rs+r)$, and $n_{..}\times rs$, respectively. In (3.75) we must necessarily have

$$n_{..} > 2rs-r \tag{3.76}$$

In addition, note that

$$\begin{aligned} H'_{ti}\, H_{ti} &= I, &\quad \text{for } i=1,2,3 \\ H'_{ti}\, H_{ti} &= \underset{\sim}{0}, &\quad \text{for } i \neq j\ . \end{aligned} \tag{3.77}$$

From (3.73) - (3.77) we can express Q_t as

$$\begin{aligned} Q_t &= \underset{\sim}{y}' H_{t1} H'_{t1}\, \underset{\sim}{y} + \underset{\sim}{y}' H_{t2} H'_{t2}\, \underset{\sim}{y} \\ &= Q_{t1} + Q_{t2}\ , \end{aligned} \tag{3.78}$$

where $Q_{t1} = \underset{\sim}{y}' H_{t1} H'_{t1}\, \underset{\sim}{y}$; $Q_{t2} = \underset{\sim}{y}' H_{t2} H'_{t2}\, \underset{\sim}{y}$.

Consider the random vector $\underset{\sim}{w}_t$ expressed as,

$$\underset{\sim}{w}_t = \underset{\sim}{\mu}_t + \left(\lambda^t_{\max} I_{r(s-1)} - L_t\right)^{1/2} H'_{t1}\underset{\sim}{y}, \qquad (3.79)$$

where $\lambda^t_{\max}$ is the largest eigenvalue of L_t. Note that the matrix $\left(\lambda^t_{\max} I_{r(s-1)} - L_t\right)^{1/2}$ is a positive semi-definite matrix; hence, it is a well-defined matrix.

Lemma 3.7

(i) $W_t \underset{\sim}{1}_{n_{..}} = W_t X_1 = W_t X_2 = W_t X_3 = \underset{\sim}{0}$.

(ii) $\operatorname{Cov}\left(\underset{\sim}{\mu}_t, H'_{t1}\underset{\sim}{y}\right) = \underset{\sim}{0}$.

Proof (i):

$$W_t \underset{\sim}{1}_{n_{..}} = \left[I_{n_{..}} - \underset{i,j}{\oplus} \left(n_{ij}^{-1} J_{n_{ij}}\right)\right]\left[\underset{i,j}{\oplus} \underset{\sim}{1}_{n_{ij}}\right]\underset{\sim}{1}_{rs}$$

$$= \left\{ \underset{i,j}{\oplus} \underset{\sim}{1}_{n_{ij}} - \left[\underset{i,j}{\oplus} \underset{\sim}{1}_{n_{ij}}\right]\right\}\underset{\sim}{1}_{rs}$$

$$= \underset{\sim}{1}_{n_{..}} - \underset{\sim}{1}_{n_{..}} = \underset{\sim}{0}_{n_{..}\times 1} .$$

$$W_t X_1 = W_t \left[\underset{i=1}{\overset{r}{\oplus}} \underset{\sim}{1}_{n_{i.}}\right] \qquad \left(n_{i.} = \sum_{j=1}^{s} n_{ij}\right)$$

$$= W_t \left[\underset{i=1}{\overset{r}{\oplus}} \left(\underset{j=1}{\overset{s}{\oplus}} \underset{\sim}{1}_{n_{ij}}\right)\underset{\sim}{1}_s\right]$$

$$= \underset{i=1}{\overset{r}{\oplus}} \underset{\sim}{1}_{n_{i.}} - \underset{i=1}{\overset{r}{\oplus}} \left[\left(\underset{j=1}{\overset{s}{\oplus}} n_{ij}^{-1} J_{n_{ij}} \underset{\sim}{1}_{n_{ij}}\right)\underset{\sim}{1}_s\right]$$

$$= \bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}} - \bigoplus_{i=1}^{r} \left[\left(\bigoplus_{j=1}^{s} \underset{\sim}{1}_{n_{ij}} \right) \underset{\sim}{1}_{s} \right]$$

$$= \bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}} - \bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}} = \underset{\sim}{0}_{n_{..} \times r} .$$

Now, the matrix $X_2 = [\bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{1j}} : \bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{2j}} : \cdots : \bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{rj}}]'$.

Furthermore,

$$\bigoplus_{i,j} n_{ij}^{-1} J_{n_{ij}} = \mathrm{Diag}\left(\bigoplus_{j=1}^{s} n_{1j}^{-1} J_{n_{1j}} , \bigoplus_{j=1}^{s} n_{2j}^{-1} J_{n_{2j}} , \ldots , \bigoplus_{j=1}^{s} n_{rj}^{-1} J_{n_{rj}} \right),$$

hence,

$W_t X_2$

$$= [I_{n_{..}} - \bigoplus_{i,j} n_{ij}^{-1} J_{n_{ij}}] X_2$$

$$= X_2 - \{ \bigoplus_{i,j} n_{ij}^{-1} J_{n_{ij}} \} X_2$$

$$= X_2 - [\bigoplus_{j=1}^{s} (n_{1j}^{-1} J_{n_{1j}} \underset{\sim}{1}_{n_{1j}})' : \bigoplus_{j=1}^{s} (n_{2j}^{-1} J_{n_{2j}} \underset{\sim}{1}_{n_{2j}})' : \ldots : \bigoplus_{j=1}^{s} (n_{rj}^{-1} J_{n_{rj}} \underset{\sim}{1}_{n_{rj}})']'$$

$$= X_2 - [\bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{1j}} : \bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{2j}} : \ldots : \bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{rj}}]'$$

$$= X_2 - X_2 = \underset{\sim}{0}_{n_{..} \times s} .$$

$$W_t X_3 = W_t [\bigoplus_{i,j} \underset{\sim}{1}_{n_{ij}}]$$

$$= X_3 - \bigoplus_{i,j} n_{ij}^{-1} J_{n_{ij}} \underset{\sim}{1}_{n_{ij}}$$

$$= X_3 - \bigoplus_{i,j} \underset{\sim}{1}_{n_{ij}}$$

$$= X_3 - X_3 = \underset{\sim}{0}.$$

(ii) We know from Lindgren (1976, p.333), Theorem 3) that $\bar{\underset{\sim}{y}}$ and the residual sums of squares, Q_t, are statistically independent. Now, $\bar{\underset{\sim}{y}} = G_t \underset{\sim}{y}$, where $G_t = \text{Diag}(n_{11}^{-1} \underset{\sim}{1}'_{n_{11}}, n_{12}^{-1} \underset{\sim}{1}'_{n_{12}}, \ldots, n_{rs}^{-1} \underset{\sim}{1}'_{n_{rs}})$. Therefore, $G_t \Sigma_t W_t = \underset{\sim}{0}$. But,

$$G_t \Sigma_t W_t = \underset{\sim}{0} \iff G_t \Sigma_t (H_{t1} H'_{t1} + H_{t2} H'_{t2}) = \underset{\sim}{0}$$

$$\Rightarrow G_t \Sigma_t (H_{t1} H'_{t1} + H_{t2} H'_{t2}) H_{t1} = \underset{\sim}{0}$$

$$\Rightarrow G_t \Sigma_t H_{t1} = \underset{\sim}{0}. \qquad \text{(see (3.77))}$$

Hence, $\text{Cov}(\underset{\sim}{\mu}_t, H'_{t1} \underset{\sim}{y}) = G_t \Sigma_t H_{t1} = \underset{\sim}{0}$. ☐

From lemma 3.7(i), since $W_t = H_{t1} H'_{t1} + H_{t2} H'_{t2}$, it follows that $H'_{t1} X_1 = 0$, $H'_{t1} X_2 = 0$, $H'_{t1} \underset{\sim}{1}_{n_{..}} = 0$. Hence,

$$E(\underset{\sim}{w}_t) = \underset{\sim}{0} \qquad (3.80)$$

and

$$\text{Var}(H'_{t1} \underset{\sim}{y}) = \sigma_e^2 I_{r(s-1)}.$$

Consequently, we have

$$\text{Var}(\underset{\sim}{w}_t) = \text{Var}(\underset{\sim}{\mu}_t) + (\lambda_{\max}^t I_{r(s-1)} - L_t)\sigma_e^2 \qquad (3.81)$$

$$= \mathrm{Diag}\left[(r\sigma^2_{\beta}+\sigma^2_{\alpha\beta})I_{s-1}, \sigma^2_{\alpha\beta}I_{(r-1)(s-1)}\right] + L_t\sigma^2_e$$

$$+ \left(\lambda^{t}_{\max}I_{r(s-1)}-L_t\right)\sigma^2_e \qquad \text{(see(3.69))}$$

$$= \mathrm{Diag}\left[\left(r\sigma^2_{\beta}+\sigma^2_{\alpha\beta}+\lambda^{t}_{\max}\sigma^2_e\right)I_{s-1},\ \left(\sigma^2_{\alpha\beta} + \lambda^{t}_{\max}\sigma^2_e\right)I_{(r-1)(s-1)}\right].$$

Hence, we can partition $\underset{\sim}{w}_t$ into $\underset{\sim}{w}_{t1}$ and $\underset{\sim}{w}_{t2}$ such that

$$\begin{aligned} &\underset{\sim}{w}_{t1} \sim N_{s-1}\left[\underset{\sim}{0},\ \left(r\sigma^2_{\beta} + \sigma^2_{\alpha\beta} + \lambda^{t}_{\max}\sigma^2_e\right)I_{s-1}\right] \\ \text{and} \\ &\underset{\sim}{w}_{t2} \sim N_{(r-1)(s-1)}\left[\underset{\sim}{0},\ \left(\sigma^2_{\alpha\beta} + \lambda^{t}_{\max}\sigma^2_e\right)I_{(r-1)(s-1)}\right] . \end{aligned} \qquad (3.82)$$

Consequently,

$$\begin{aligned} &Q_{t1} = \underset{\sim}{w}'_{t1}\underset{\sim}{w}_{t1}/\left(r\sigma^2_{\beta} + \sigma^2_{\alpha\beta} + \lambda^{t}_{\max}\sigma^2_e\right) \sim \chi^2_{s-1} \\ \text{and} \\ &Q_{t2} = \underset{\sim}{w}'_{t2}\underset{\sim}{w}_{t2}/\left(\sigma^2_{\alpha\beta} + \lambda^{t}_{\max}\sigma^2_e\right) \sim \chi^2_{(r-1)(s-1)} . \end{aligned} \qquad (3.83)$$

Furthermore, from (3.81), Q_{t1} and Q_{t2} are independently distributed; hence,

$$F_{t1} = \frac{\underset{\sim}{w}_{t1}' \underset{\sim}{w}_{t1} / (s-1)}{\underset{\sim}{w}_{t2}' \underset{\sim}{w}_{t2} / (r-1)(s-1)} , \qquad (3.84)$$

can be used to test the hypothesis $H_o : \sigma^2_\beta = 0$ vs. $H_a : \sigma^2_\beta \neq 0$. Under H_o, F_{t1} has an F distribution with numerator and denominator degrees of freedom equal to s-1 and (r-1)(s-1), respectively. We reject the H_o at the α-level of significance if $F_{t1} > F_{\alpha,(s-1),(r-1)(s-1)}$, that is, if F_{t1} exceeds the upper α100% point of the corresponding F distribution.

Theorem 3.7

The vector $\underset{\sim}{w}_t$ and the random variable Q_{t2} are independently distributed.

Proof: Recall from (3.78) that $Q_t = Q_{t1} + Q_{t2}$, where $Q_{t1} = \underset{\sim}{y}' H_{t1} H_{t1}' \underset{\sim}{y}$ and $Q_{t2} = \underset{\sim}{y}' H_{t2} H_{t2}' \underset{\sim}{y}$. We know that $\bar{\underset{\sim}{y}}$ is independently distributed from Q_{t2}; hence, $\underset{\sim}{\mu}_t = PC_t \bar{\underset{\sim}{y}}$ is independently distributed from Q_{t2}.

Furthermore, $H_{t1}' \text{Var}(\underset{\sim}{y}) H_{t2} H_{t2}' = H_{t1}' \left(X_2 X_2' \sigma^2_\beta + X_3 X_3' \sigma^2_{\alpha\beta} + I\sigma^2_e \right) H_{t2}$

$$= H_{t1}' H_{t2} \sigma^2_e \qquad \text{(see lemma 3.7(i))}$$

$$= \underset{\sim}{0} \qquad \text{(see (3.77))}.$$

But this implies that $H'_{t1}\underset{\sim}{y}$ and Q_{t2} are independently distributed (Searle 1971, p.59, Thm. 3). Since Q_{t2} is independently distributed from $\underset{\sim}{\mu}_t$ and $H'_{t1}\underset{\sim}{y}$, we have that Q_{t2} is independently distributed from $\underset{\sim}{w}_t$ (see (3.79)). ☐

Consequently, as a result of Theorem 3.7, Q_{t2} is independently distributed from $\underset{\sim}{w}_{t2}$.

Theorem 3.8

$$Q_{t2}/\sigma_e^2 \sim \chi^2_{n_{\cdot\cdot}-2rs+r} .$$

Proof: $Q_{t2} = \underset{\sim}{y}'H_{t2}H'_{t2}\underset{\sim}{y}$. We know from lemma 3.7 and (3.71) that $\sigma_e^{-1/2}H_{t2}H'_{t2}\text{Var}(\underset{\sim}{y}) = H_{t2}H'_{t2}$.

But, $H_{t2}H'_{t2} = H_{t2}H'_{t2}H_{t2}H_{t2}$, since $H'_{t2}H_{t2} = I_{n_{\cdot\cdot}}$. That is, $H_{t2}H'_{t2}$ is idempotent. Furthermore, $\text{rank}(H_{t2}) = n_{\cdot\cdot}-2rs+r$; hence, from Searle (1971, p.57 Theorem 2), $Q_{t2}/\sigma_e^2 \sim \chi^2_{n_{\cdot\cdot}-2rs+r}$. ☐

From (3.83) and theorems 3.7 and 3.8,

$$F_{t2} = \frac{\underset{\sim}{w}'_{t2}\,\underset{\sim}{w}_{t2} \;/\; (r-1)(s-1)}{\lambda^t_{\max}Q_{t2}/(n_{\cdot\cdot}-2rs+r)} \qquad (3.85)$$

can be used to test the hypothesis $H_o: \sigma^2_{\alpha\beta} = 0$ vs. $H_a: \sigma^2_{\alpha\beta} \neq 0$. Note that under H_o, F_{t2} has an F distribution with numerator and denominator degrees of freedom equal to $(r-1)(s-1)$ and $n_{..}-2rs+r$, respectively. We reject H_o at the α-level of significance if $F_{t2} > F_{\alpha,(r-1)(s-1),(n..-2rs+r)}$.

3.4.3 Derivation of an Exact Test for the Fixed Effects

Consider again model (3.62) with expectation vector and variance-covariance matrix defined in (3.63) and (3.64), respectively. Let $A_1 = B_1B_1'$, i.e., $A_1 = I_r \times J_s$. The matrices A_1 and A_2 (recall $A_2 = B_2B_2' = J_r \times I_s$) can easily be shown to commute, i.e., $A_1A_2 = A_2A_1$. Hence, from Searle (1984, p. 312), there exists, an orthogonal matrix P^* which simultaneously diagonalizes the two matrices. Let $D_1^* = P^*A_1P^{*'}$ and $D_2^* = P^*A_2P^{*'}$. The matrices D_1^* and D_2^* are diagonal matrices whose diagonal elements are the eigenvalues of A_1 and A_2, respectively. Now as pointed out in Thomsen (1975):

(i) The matrix A_1 has eigenvalues equal to s and zero with multiplicities r and rs-r, respectively,

(ii) The matrix A_2 has eigenvalues equal to r and zero with multiplicities s and rs-s, respectively,

and (iii) The $\text{rank}(A_1+A_2) = r+s-1$. (3.86)

Lemma 3.8

The first row of P^* can be chosen to be $(rs)^{-1/2}\ \underset{\sim}{1}'_{rs}$.

Proof: We need to show that the vector $\underset{\sim}{1}_{rs}$ is an eigenvector of both A_1 and A_2 corresponding to eigenvalues s and r respectively. Note that

$$\begin{aligned} A_1\underset{\sim}{1}_{rs} &= (I_r \times J_s)\underset{\sim}{1}_{rs} \\ &= \underset{\sim}{1}_r \times s\underset{\sim}{1}_s \\ &= s\underset{\sim}{1}_{rs} \ . \end{aligned}$$

Since, s is an eigenvalue of A_1 we must have that, $\underset{\sim}{1}_{rs}$ is an eigenvector of A_1 with eigenvalue s. Similarily,

$$\begin{aligned} A_2\underset{\sim}{1}_{rs} &= (J_r \times I_s)\underset{\sim}{1}_{rs} \\ &= r\underset{\sim}{1}_r \times \underset{\sim}{1}_s \\ &= r\underset{\sim}{1}_{rs} \end{aligned}$$

and r is an eigenvalue of A_2, hence, $\underset{\sim}{1}_{rs}$ is a corresponding eigenvector of A_2. Consequently, $(rs)^{-1/2}\ \underset{\sim}{1}_{rs}$ is an eigenvector of both A_1 and A_2. □

Consider the tranformation

$$\underset{\sim}{u}^* = P^*\bar{\underset{\sim}{y}} \ . \qquad (3.87)$$

The matrix P^* can be partitioned into

$$P^* = [(rs)^{-1/2}\, \underset{\sim}{1}'_{rs} : P_2^{*\prime} : P_3^{*\prime} : P_4^{*\prime}]', \qquad (3.88)$$

where P_2^* consists of the r-1 rows of P^* such that $P_2^* A_1 P_2^{*\prime} = sI_{r-1}$ and $P_2^* A_2 P_2^{*\prime} = \underset{\sim}{0}_{(r-1)\times(r-1)}$; P_3^* consists of the s-1 rows of P^* such that $P_3^* A_1 P_3^* = \underset{\sim}{0}_{(s-1)\times(s-1)}$ and $P_3^* A_2 P_3^{*\prime} = rI_{s-1}$; P_4^* consists of the (r-1)(s-1) rows of P^* such that $P_4^* A_1 P_4^{*\prime} = \underset{\sim}{0}_{(r-1)\times(s-1)}$ and $P_4^* A_2 P_4^{*\prime} = \underset{\sim}{0}_{(r-1)\times(s-1)}$. Consequently, we can partition $\underset{\sim}{u}^*$ into $\underset{\sim}{u}^* = [u_1^*, \underset{\sim}{u}_2^{*\prime}, \underset{\sim}{u}_3^{*\prime}, u_4^{*\prime}]'$, where

$$u_1^* \sim N_1\Big(\mu(rs)^{1/2} + \underset{\sim}{1}'_{rs} B_1 \underset{\sim}{\alpha},\ r\sigma^2_\beta + \sigma^2_{\alpha\beta} + (rs)^{-1/2}\big[\sum_{i,j} n_{ij}^{-1}\big]\sigma^2_e\Big),$$

$$\underset{\sim}{u}_2^* \sim N_{r-1}\big(P_2^* B_1 \underset{\sim}{\alpha},\ \sigma^2_{\alpha\beta} I_{r-1} + P_2^* K P_2^{*\prime} \sigma^2_e\big), \qquad (3.89)$$

$$\underset{\sim}{u}_3^* \sim N_{s-1}\big(\underset{\sim}{0},\ (r\sigma^2_\beta + \sigma^2_{\alpha\beta}) I_{s-1} + P_3^* K P_3^{*\prime} \sigma^2_e\big),$$

$$\underset{\sim}{u}_4^* \sim N_{(r-1)(s-1)}\big(\underset{\sim}{0},\ \sigma^2_{\alpha\beta} I_{(r-1)(s-1)} + P_4^* K P_4^{*\prime} \sigma^2_e\big).$$

Recall from (3.72) and (3.73) that

$$Q_t = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2 = \underset{\sim}{y}' W_t \underset{\sim}{y}, \text{ where } W_t = I_{n_{..}} - \bigoplus_{i,j} (J_{n_{ij}}/n_{ij}).$$

Now instead of using the partitioning defined in (3.75), consider the following

$$\Lambda_4 = \mathrm{Diag}\left(I_{\nu_1}, I_{\nu_2}, \underset{\sim}{0}_{rs\times rs}\right) \tag{3.90}$$

and

$$H_F = \left[H_{F1} : H_{F2} : H_{F3}\right],$$

where $\nu_1 = (r-1) + (r-1)(s-1) = s(r-1)$; $\nu_2 = n_{\bullet\bullet} - 2rs + s$. The matrices H_{F1}, H_{F2}, and H_{F3} are matrices of orders $n_{\bullet\bullet}\times(s(r-1))$. $n_{\bullet\bullet}\times(n_{\bullet\bullet}-2rs+s)$, and $n_{\bullet\bullet}\times rs$, respectively. We require that

$$n_{\bullet\bullet} > 2rs-s . \tag{3.91}$$

In order to satisfy both (3.76) and (3.91) we will need

$$n_{\bullet\bullet} > \max\left[2rs - r, 2rs - s\right] . \tag{3.92}$$

Consider the random vector $\underset{\sim}{w}^*$ of order $[s(r-1)]\times 1$ defined as

$$\underset{\sim}{w}^* = \underset{\sim}{u}^*_{2:4} + \left(\lambda^F_{\max} I_{v_1} - L_{2:4}\right)^{1/2} H'_{F1}\, \underset{\sim}{y}, \tag{3.93}$$

where

$$\underset{\sim}{u}^*_{2:4} = \left[\underset{\sim}{u}^{*\prime}_2 : \underset{\sim}{u}^{*\prime}_4\right]';$$

$$L_{2:4} = \begin{bmatrix} P_2^* \\ P_4^* \end{bmatrix} K \, [P_2^* \; : \; P_4^*\,] ;$$

$$= \begin{bmatrix} P_2^* K P_2^{*\prime} & P_2^* K P_4^{*\prime} \\ P_4^* K P_2^{*\prime} & P_4^* K P_4^{*\prime} \end{bmatrix} \quad ;$$

$\lambda_{\max}^{F}$ is the largest eigenvalue of $L_{2:4}$.

Lemma 3.9

$$E(\underset{\sim}{u}^*_{2:4}) = \begin{bmatrix} P_2^* \, B_1\underset{\sim}{\alpha} \\ \underset{\sim}{0} \end{bmatrix}$$

and

$$\operatorname{Var}(\underset{\sim}{u}^*_{2:4}) = \sigma^2_{\alpha\beta} I_{s(r-1)} + L_{2:4}\sigma^2_e$$

where $\underset{\sim}{0}$ is an $(r-1)(s-1)\times 1$ vector of zeroes.

Proof: Let $P^*_{2:4} = [P_2^{*\prime} \; : \; P_4^{*\prime}\,]'$. We have

$$E(\underset{\sim}{u}^*_{2:4}) = E(P^*_{2:4} \, \bar{\underset{\sim}{y}}\,)$$

$$= P^*_{2:4} \, E(\bar{\underset{\sim}{y}})$$

$$= P^*_{2:4} \, (\mu \underset{\sim}{1}_{rs} + B_1\underset{\sim}{\alpha})$$

$$= \begin{bmatrix} P_2^{*} B_1 \underset{\sim}{\alpha} \\ \underset{\sim}{0} \end{bmatrix}$$

$$\mathrm{Var}(\underset{\sim}{u}_{2:4}^{*}) = P_{2:4}^{*} \{ \mathrm{Var}\ \underset{\sim}{\bar{y}} \} P_{2:4}^{*\prime}$$

$$= P_{2:4}^{*} \{ A_2 \sigma_{\beta}^{2} + I\sigma_{\alpha\beta}^{2} + K\sigma_{e}^{2} \} P_{2:4}^{*\prime}$$

$$= \begin{bmatrix} \Sigma_{22} & \Sigma_{24} \\ \Sigma_{24}^{\prime} & \Sigma_{44} \end{bmatrix}$$

where

$$\Sigma_{22} = \mathrm{Var}(\underset{\sim}{u}_2^{*}) = \sigma_{\alpha\beta}^{2} I_{r-1} + P_2^{*} K P_2^{*\prime} \sigma_e^{2}$$

$$\Sigma_{44} = \mathrm{Var}(\underset{\sim}{u}_4^{*}) = \sigma_{\alpha\beta}^{2} I_{(r-1)(s-1)} + P_4^{*} K P_4^{*\prime} \sigma_e^{2} ,$$

and $\Sigma_{24} = \mathrm{Cov}(\underset{\sim}{u}_2^{*} , \underset{\sim}{u}_4^{*})$

$$= P_2^{*} A_2 P_4^{*\prime} \sigma_{\beta}^{2} + P_2^{*} K P_4^{*\prime} \sigma_e^{2}.$$

But, $P_2^{*} A_2 P_2^{*\prime} = \underset{\sim}{0} \iff P_2^{*} B_2 B_2^{\prime} P_4^{*\prime} = \underset{\sim}{0}$

$$\iff P_2^{*} B_2 = \underset{\sim}{0};$$

hence, $P_2^{*} A_2 = \underset{\sim}{0}$. Consequently, $\Sigma_{24} = P_2^{*} K P_4^{*\prime} \sigma_e^{2}$ and therefore,

$$\mathrm{Var}(\underset{\sim}{u}^{*}_{2:4}) = \begin{bmatrix} \mathrm{Var}(\underset{\sim}{u}^{*}_{2}) & P^{*}_{2}KP^{*\prime}_{4}\,\sigma^2_e \\ P^{*}_{4}KP^{*\prime}_{2}\,\sigma^2_e & \mathrm{Var}(\underset{\sim}{u}^{*}_{4}) \end{bmatrix}$$

$$= \sigma^2_{\alpha\beta} I_{(s(r-1))} + \begin{bmatrix} P^{*}_{2}KP^{*\prime}_{2} & P^{*}_{2}KP^{*\prime}_{4} \\ P^{*}_{4}KP^{*}_{2} & P^{*}_{4}KP^{*\prime}_{4} \end{bmatrix} \sigma^2_e$$

$$= \sigma^2_{\alpha\beta} I_{s(r-1)} + L_{2:4}\,\sigma^2_e \,. \quad \square$$

Theorem 3.9

$$E(\underset{\sim}{w}^{*}) = \begin{bmatrix} P^{*}_{2} B_1 \underset{\sim}{\alpha} \\ \underset{\sim}{0} \end{bmatrix} \quad \text{and} \quad \mathrm{Var}(\underset{\sim}{w}^{*}) = \left(\sigma^2_{\alpha\beta} + \lambda^F_{\max}\,\sigma^2_e\right) I_{s(r-1)} \,.$$

Proof: $E(\underset{\sim}{w}^{*}) = E(\underset{\sim}{u}_{2:4}) + \left(\lambda^F_{\max} I_{\nu_1} - L_{2:4}\right)^{1/2} H'_{F1} E(\underset{\sim}{y})$

$$= E(\underset{\sim}{u}^{*}_{2:4}) + \left(\lambda^F_{\max} I_{\nu_1} - L_{2:4}\right)^{1/2} H'_{F1} \left(u \underset{\sim}{1}_{n_{..}} + X_1 \underset{\sim}{\alpha}\right) .$$

Now as in (3.78), using (3.90), we can express $Q_t = \underset{\sim}{y}' W_t \underset{\sim}{y}$ as

$$Q_t = \underset{\sim}{y}'\left(H_{F1}H'_{F1} + H_{F2}H'_{F2}\right)\underset{\sim}{y},$$

that is, $W_t = H_{F1}H'_{F1} + H_{F2}H'_{F2}$. Furthermore,

$$H'_{Fi}\,H_{Fi} = I, \qquad \text{for } i=1,2,3$$

and

$$H'_{Fi}\,H_{Fj} = \underset{\sim}{0}, \qquad \text{for } i \neq j \;;$$

hence, applying lemma 3.7, we have that $H'_{F1}\,\underset{\sim}{1}_N = H'_{F1}X_1 = \underset{\sim}{0}$. Consequently,

$$\begin{aligned} E(\underset{\sim}{w}^*) &= E(\underset{\sim}{u}^*_{2:4}) \\ &= P^*_{2:4}B_1\underset{\sim}{\alpha} \qquad\qquad (P^*_{2:4} = [P^{*\prime}_2 : P^{*\prime}_4]^\prime) \\ &= \begin{bmatrix} P^*_2B_1\alpha \\ 0 \end{bmatrix} \qquad\qquad (\text{since } P^*_4A_1 = 0 \Rightarrow P^*_4B_1 = 0) \end{aligned} \tag{3.94}$$

We also have that the $\mathrm{Cov}(\underset{\sim}{u}^*_{2:4}\,,\ H'_{F1}\,\underset{\sim}{y}\,) = \underset{\sim}{0}$, which follows from lemma 3.7(ii), by substituting H_{F1} and H_{F2} for H_{T1} and H_{T2}, respectively. Hence,

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{w}^*) &= \mathrm{Var}(\underset{\sim}{u}^*_{2:4}) + (\lambda^F_{\max}I_{u_1} - L_{2:4})^{1/2}H'_{F1}\mathrm{Var}(\underset{\sim}{y})H_{F1}(\lambda^F_{\max}I_{u_1} - L_{2:4})^{1/2} \\ &= \mathrm{Var}(u^*_{2:4}) + (\lambda^F_{\max}I_{u_1} - L_{2:4})^{1/2}H'_{F1} \\ &\quad \times [X_2X'_2\sigma^2_\beta + X_3X'_3\sigma^2_{\alpha\beta} + I\sigma^2_e]H_{F1}(\lambda^F_{\max}I_{u_1} - L_{2:4})^{1/2} \,. \end{aligned}$$

Again, from lemma 3.7(ii) we have that $H'_{F1}X_2 = H'_{F1}X_3 = \underset{\sim}{0}$.

Hence,

$$\begin{aligned}\mathrm{Var}(\underset{\sim}{w}^*) &= \mathrm{Var}(\underset{\sim}{u}^*_{2:4}) + (\lambda^F_{\max} I_{u_1} - L_{2:4})\sigma^2_e \\ &= \sigma^2_{\alpha\beta} I_{s(r-1)} + L_{2:4}\sigma^2_e + (\lambda^F_{\max} I_{u_1} - L_{2:4})\sigma^2_e \\ &= (\sigma^2_{\alpha\beta} + \lambda^F_{\max}\sigma^2_e) I_{s(r-1)} \,. \end{aligned} \qquad (3.95)$$

□

The random vector $\underset{\sim}{w}^*$ can be expressed as

$$\underset{\sim}{w}^* = P^*_{2:4}B_1\underset{\sim}{\alpha} + P^*_{2:4}(\underset{\sim}{\alpha\beta}) + P^*_{2:4}\underset{\sim}{\bar{e}} + (\lambda^F_{\max} I_{u_1} - L_{2:4})^{1/2} H'_{F1}\,\underset{\sim}{e}. \qquad (3.96)$$

(Note that $P^*_{2:4}B_2\underset{\sim}{\beta} = \underset{\sim}{0}$, since $P^*_2 B_2 = P^*_4 B_2 = \underset{\sim}{0}$ from (3.88) and the fact that $P^*_2 A_2 P^{*\prime}_2 = \underset{\sim}{0}$ and $P^*_4 A_2 P^{*\prime}_4 = \underset{\sim}{0}$.) Let X^* and $\underset{\sim}{\varepsilon}^*$ be defined as

$$\begin{aligned} X^* &= P^*_{2:4}B_1 \\ \underset{\sim}{\varepsilon}^* &= P^*_{2:4}(\underset{\sim}{\alpha\beta}) + P^*_{2:4}\underset{\sim}{\bar{e}} + (\lambda^F_{\max} I_{u_1} - L_{2:4})^{1/2} H'_{F1}\,\underset{\sim}{e}, \end{aligned} \qquad (3.97)$$

respectively. Note that $\underset{\sim}{\varepsilon}^* \sim N[\underset{\sim}{0}, (\sigma^2_{\alpha\beta} + \lambda^F_{\max}\sigma^2_e) I_{s(r-1)}]$.

From (3.96) we have

$$\underset{\sim}{w}^{*} = X^{*}\underset{\sim}{\alpha} + \underset{\sim}{\varepsilon}^{*} \qquad (3.98)$$

The generalized least squares normal equations for obtaining the estimates of the elements in $\underset{\sim}{\alpha}$ of (3.98) are

$$\begin{aligned} &X^{*\prime}\left[\mathrm{Var}(\underset{\sim}{w}^{*})\right]^{-1}X^{*}\underset{\sim}{\alpha}^{o} = X^{*\prime}\left[\mathrm{Var}(\underset{\sim}{w}^{*})\right]^{-1}\underset{\sim}{w}^{*} \\ &\Leftrightarrow X^{*\prime}X^{*}\underset{\sim}{\alpha}^{o} = X^{*\prime}\underset{\sim}{w}^{*} \end{aligned} \qquad (3.99)$$

since $\mathrm{Var}(\underset{\sim}{w}^{*}) = \left(\sigma^2_{\alpha\beta} + \lambda^{F}_{\max}\sigma^2_{e}\right)I_{s(r-1)}$. Solving (3.99) for $\underset{\sim}{\alpha}^{o}$ we obtain

$$\underset{\sim}{\alpha}^{o} = \left(X^{*\prime}X^{*}\right)^{-} X^{*\prime}\underset{\sim}{w}^{*} . \qquad (3.100)$$

Consider testing

$$H_o:\ R^{*\prime}\underset{\sim}{\alpha} = \underset{\sim}{m}^{*} , \qquad (3.101)$$

where $R^{*\prime}$ is a full row-rank matrix and $R^{*\prime}\underset{\sim}{\alpha}$ is a set of estimable linear functions of the elements of $\underset{\sim}{\alpha}$; $\underset{\sim}{m}^{*}$ is a vector, of order (g×1) (g equals the row rank of $R^{*\prime}$), of specified constants. Let

$$\underset{\sim}{\ell}^{*}_{a} = R^{*\prime}\underset{\sim}{\alpha}^{o} - \underset{\sim}{m}^{*} . \qquad (3.102)$$

From (3.100) and the fact that $R^{*}\underset{\sim}{\alpha}$ is an estimable linear function of $\underset{\sim}{\alpha}$, we have that

$$\begin{aligned} E(\underset{\sim}{\ell}_a^{*}) &= R^{*\prime}\underset{\sim}{\alpha} - \underset{\sim}{m}^{*} \\ \mathrm{Var}(\underset{\sim}{\ell}_a^{*}) &= R^{*\prime} G_{X^{*}} R^{*}(\sigma^2_{\alpha\beta} + \lambda^{F}_{\max}\sigma^2_e), \end{aligned} \qquad (3.103)$$

where $G_{X^{*}} = (X^{*\prime}X^{*})^{-}$.

It follows from Searle (1971, p. 57, Theorem 2) that

$$\begin{aligned} Q_\alpha^{*} &= \underset{\sim}{\ell}_a^{*\prime}(R^{*\prime}G_{X^{*}}R^{*})^{-1}\,\underset{\sim}{\ell}_a^{*}/\sigma^{*2} \\ &\sim \chi^{2\prime}[g,\ (R^{*\prime}\underset{\sim}{\alpha}-\underset{\sim}{m}^{*})(R^{*\prime}G_{X^{*}}R^{*})^{-1}(R^{*\prime}\underset{\sim}{\alpha}-\underset{\sim}{m}^{*})/2\sigma^{*2}\ ], \end{aligned} \qquad (3.104)$$

where $\sigma^{*2} = \sigma^2_{\alpha\beta} + \lambda^{F}_{\max}\sigma^2_e$. Now consider

$$\begin{aligned} SSE^{*} &= \underset{\sim}{w}^{*\prime}\underset{\sim}{w}^{*} - \underset{\sim}{\alpha}^{o\prime}X^{*\prime}\underset{\sim}{w}^{*} \\ &= \underset{\sim}{w}^{*\prime}\,(I_{s(r-1)} - X^{*}(X^{*\prime}X^{*})^{-}X^{*\prime})\underset{\sim}{w}^{*}\ . \end{aligned} \qquad (3.105)$$

Theorem 3.10

$$SSE^*/\sigma^{*2} \sim \chi^2_{(r-1)(s-1)} .$$

Proof: The matrix $\left(I_{s(r-1)} - X^*(X^{*\prime}X^*)^- X^{*\prime}\right)$ is idempotent; hence,

$$\begin{aligned} \text{rank}\left(I_{s(r-1)} - X^*(X^{*\prime}X^*)^- X^{*\prime}\right) \\ &= \text{trace}\left(I_{s(r-1)} - X^*(X^{*\prime}X^*)^- X^{*\prime}\right) \\ &= \text{rank}\left(I_{s(r-1)}\right) - \text{rank}\left(X^*\right) \\ &= s(r-1) - \text{rank}(X^*). \end{aligned}$$

But, $\text{rank}(X^*) = \text{rank}\left(P^*_{2:4}B_1\right)$

$$\begin{aligned} &= \text{rank}\left(P^*_{2:4}A_1P^{*\prime}_{2:4}\right) \\ &= \text{rank}\left\{\text{Diag}\left(sI_{r-1}, \underset{\sim}{0}_{(r-1)(s-1)}\right)\right\} \\ &= r-1. \end{aligned}$$

Hence, $\text{rank}\left(I_{s(r-1)}-X^*(X^{*\prime}X^*)^- X^{*\prime}\right) = s(r-1)-(r-1) = (r-1)(s-1)$. Furthermore, $\left[E(\underset{\sim}{w}^*)\right]^\prime\left[I_{s(r-1)}-X^*(X^{*\prime}X^*)^- X^{*\prime}\right]\left[E(\underset{\sim}{w}^*)\right] = \underset{\sim}{0}$. Therefore, from Searle (1971, p.57, Theorem 2),

$$SSE^*/\sigma^{*2} \sim \chi^2_{(r-1)(s-1)}. \qquad \square$$

It can easily be shown that Q^*_α in (3.104) and SSE^* in (3.105) are independently distributed (see analogous proof in Theorem 3.4). Hence, from (3.104) and Theorem 3.10, we have that

$$F^*(H) = \frac{\underset{\sim}{\ell}_a^{*\prime}(R^{*\prime} G_{X^*} R^*)^{-1}\, \underset{\sim}{\ell}_a^*/g}{SSE^*/(r-1)(s-1)} \tag{3.106}$$

$$\sim F'[g,\ (r-1)(s-1),\ (R^*\underset{\sim}{\alpha}-\underset{\sim}{m}^*)(R^{*\prime} G_{X^*} R^*)^{-1}(R^*\underset{\sim}{\alpha}-\underset{\sim}{m}^*)/2\sigma^{*2}].$$

Under the hypothesis, H_o: $R^{*\prime}\underset{\sim}{\alpha} = \underset{\sim}{m}^*$, the noncentrality parameter is zero and $F^*(H) \sim F_{g,(r-1)(s-1)}$, i.e., $F^*(H)$ has a central F distribution. Therefore, $F^*(H)$ provides an exact test for H_o. In particular we can test the hypothesis H_o: $\alpha_1 = \alpha_2 = \cdots = \alpha_r$.

In the following table, three machines (the fixed effects) in a company are to be compared in terms of their speed in using a particular component. Four employees (the random effects) are randomly selected to participate in the experiment. Each employee is to operate the machine three different times and an overall score which takes into account the number and quality of components produced is recorded. At the end of the study we are left with the following data:

Table 3.2 Example of an Unbalanced Two-Way Cross Classification Mixed Design (Source: Miliken and Johnson (1984, Table 23.1, p. 285))

Machine	Person	time 1	time 2	time 3
1	1	52.0		
1	2	51.8	52.8	
1	3	60.0		
1	4	51.1	52.3	
2	1			64.0
2	2	59.7	60.0	59.0
2	3	68.6	65.8	
2	4	63.2	62.89	62.2
3	1	67.5	67.2	66.9
3	2	61.5	61.7	62.3
3	3	70.8	70.6	71.0
3	4	64.1	66.2	64.0

Note that we have an unbalanced design (some of the data has been randomly deleted to illustrate this design). Using the program in Appendix F, we test the fixed main effect and the variance components associated with the random effects (main and interaction effect).

Table 3.2 - continued

Results:

For testing $H_o : \sigma^2_{\alpha\beta}/\sigma^2_e = 0$

test statistic = 45.70 with numerator df = 6,

denominator df = 6

p-value $< .0001$

For testing $H_o : \sigma^2_{\beta}/\sigma^2_e = 0$

test statistic = 1.94 with numerator df = 3,

denominator df = 6

p-value = .224

For testing the fixed effects: H_o: $\alpha_1 = \alpha_2 = \alpha_3$ vs H_a: not H_o

test statistic = 12.09 with numerator df = 2,

denominator df = 6

p-value = .008.

Since the interaction appears to be significant, we have to analyze the fixed effects for each level of the random effects.

3.4.4 Reduction of the Exact Tests When the Design is Balanced

In this section we investigate the situation where model (3.60) is defined for $i=1,2,\ldots,r$; $j=1,2,\ldots,s$; $k=1,2,\ldots,n$, that is, we have a balanced design. We need to show that the derived exact tests reduce to the conventional test statistics employed under a balanced situation.

A. Reduction of the Exact Tests for Testing the Variance Component associated with the Random Main Effect

Recall from Section 3.4.1 that for testing the H_o: $\sigma_\beta^2 = 0$ vs. H_a: $\sigma_\beta^2 \neq 0$, we derived the test statistic defined in (3.84),

$$F_{t1} = \frac{\mathbf{w}_{t1}'\mathbf{w}_{t1}/(s-1)}{\mathbf{w}_{t2}'\mathbf{w}_{t2}/(r-1)(s-1)} .$$

From Khuri (1982), the corresponding conventional F-test employed when the design is balanced can be written as

$$F_3 = \frac{\mathbf{y}'P_2\mathbf{y}/(s-1)}{\mathbf{y}'P_3\mathbf{y}/(r-1)(s-1)}, \qquad (3.107)$$

where

$$P_3 = n^{-1}(I_{rs}\times J_n) - \frac{1}{sn}(I_r\times J_{sn}) - \frac{1}{rn}(J_r\times I_s\times J_n) + \frac{1}{rsn} J_{rsn} \qquad (3.108)$$

and

$$P_2 = \frac{1}{rn}(J_r \times I_s \times J_n) - \frac{1}{rsn} J_{rsn}.$$

Theorem 3.11

$F_{t1} = F_3$ when we have a balanced design.

Proof:

From (3.79),

$$\underset{\sim}{w}_t = \underset{\sim}{\mu}_t + (\lambda^t_{\max} I_{r(s-1)} - L_t)^{1/2} H'_{t1}\underset{\sim}{y}.$$

Now $L_t = PC_tKC'_tP'$, where $K = \mathrm{Diag}(n^{-1}_{11}, n^{-1}_{12}, \ldots, n^{-1}_{rs})$.

When we have a balanced design $n_{ij} = n \;\forall_{ij}$, hence, the matrix K reduces to $n^{-1}I_{rs}$. Now, recall that $\lambda^t_{\max}$ is the largest eigenvalue of L_t, but, for the balanced design,
$L_t = PC_t\{n^{-1}I_{rs}\}C'_tP' = n^{-1}PC_tC'_tP' = n^{-1}I_{rs}$.
Hence, $\lambda^t_{\max} = n^{-1}$. Consequently, when we have a balanced design,

$$\underset{\sim}{w}_t = \underset{\sim}{\mu}_t = PC_t\bar{\underset{\sim}{y}} . \tag{3.109}$$

It follows from (3.82) that $\underset{\sim}{w}_t$ can be partitioned into

$$\underset{\sim}{w}_{t1} = P^{(1)}C_t\bar{\underset{\sim}{y}} \text{ and } \underset{\sim}{w}_{t2} = P^{(2)}C_t\bar{\underset{\sim}{y}}$$

where $P^{(1)}$ consists of the s-1 rows of P such that $P^{(1)}C_tA_2C_t'P^{(1)'} = rI_{s-1}$ and $P^{(2)}$ consists of the remaining (r-1)(s-1) rows of P such that $P^{(2)}C_tA_2C_t'P^{(2)'} = \underset{\sim}{Q}$ (the matrix $\underset{\sim}{Q}$ is of order $[(r-1)(s-1)]\times[(r-1)(s-1)]$.) Note that

$$C_tA_2C_t' = (I_r \times T)(J_r \times I_s)(I_r \times T') \qquad \text{(T defined in (3.66))}$$

$$= (J_r \times T)(I_r \times T')$$

$$= J_r \times TT' = J_r \times I_{s-1}.$$

Furthermore, consider the matrix $(\frac{1}{\sqrt{r}}\underset{\sim}{1}_r' \times I_{s-1})$. We have that

$$(\frac{1}{\sqrt{r}}\underset{\sim}{1}_r' \times I_{s-1})C_tA_2C_t'(\frac{1}{\sqrt{r}}\underset{\sim}{1}_r \times I_{s-1}) = rI_{s-1};$$

hence, $P^{(1)}$ can be chosen to be the matrix $(\frac{1}{\sqrt{r}}\underset{\sim}{1}_r' \times I_{s-1})$. Consequently,

$$\underset{\sim}{w}_{t1}'\underset{\sim}{w}_{t1} = \bar{\underset{\sim}{y}}'C_t'P^{(1)'}P^{(1)}C_t\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'(I_r \times T')(\frac{1}{\sqrt{r}}\underset{\sim}{1}_r \times I_{s-1})(\frac{1}{\sqrt{r}}\underset{\sim}{1}'_r \times I_{s-1})(I_r \times T)\bar{\underset{\sim}{y}}$$

$$= \frac{1}{r}\bar{\underset{\sim}{y}}'(I_r \times T')(J_r \times I_{s-1})(I_r \times T)\bar{\underset{\sim}{y}}$$

$$= \frac{1}{r}\bar{\underset{\sim}{y}}'[J_r \times T'T]\bar{\underset{\sim}{y}}$$

$$= \frac{1}{r}\bar{\underset{\sim}{y}}'[J_r \times (I_s - \frac{1}{s}J_s)]\bar{\underset{\sim}{y}} \qquad \text{(see Appendix D)}$$

$$= \bar{\underset{\sim}{y}}'[\frac{1}{r}(J_r \times I_s) - \frac{1}{rs}(J_r \times J_s)]\bar{\underset{\sim}{y}}$$

$$= n^{-2}\underset{\sim}{y}'(I_{rs} \times \underset{\sim}{1}_n)[\frac{1}{r}(J_r \times I_s) - \frac{1}{rs}(J_r \times J_s)](I_{rs} \times \underset{\sim}{1}'_n)\underset{\sim}{y}$$

$$= \frac{1}{n}\underset{\sim}{y}'[\frac{1}{rn}(J_r \times I_s \times J_n) - \frac{1}{rsn}J_{rsn}]\underset{\sim}{y}$$

$$= n^{-1}\underset{\sim}{y}'P_2\underset{\sim}{y}.$$

Now,

$$\underset{\sim}{w}_t = \begin{bmatrix} \underset{\sim}{w}_{t1} \\ \underset{\sim}{w}_{t2} \end{bmatrix} ;$$

hence,

$$\underset{\sim}{w}'_t\underset{\sim}{w}_t = \underset{\sim}{w}'_{t1}\underset{\sim}{w}_{t1} + \underset{\sim}{w}'_{t2}\underset{\sim}{w}_{t2}, \quad \text{that is,}$$

$$\underset{\sim}{w}'_{t2}\underset{\sim}{w}_{t2} = \underset{\sim}{w}'_t\underset{\sim}{w}_t - \underset{\sim}{w}'_{t1}\underset{\sim}{w}_{t1}.$$

But,

$$\underset{\sim}{w}_t'\underset{\sim}{w}_t = \bar{\underset{\sim}{y}}'C_t'P'PC_t\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'C_t'C_t\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'(I_r \times T')(I_r \times T)\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'(I_r \times T'T)\bar{\underset{\sim}{y}}$$

$$= n^{-2}\underset{\sim}{y}'(I_{rs} \times \underset{\sim}{1}_n)(I_r \times T'T)(I_{rs} \times \underset{\sim}{1}_n')\underset{\sim}{y}$$

$$= n^{-2}\underset{\sim}{y}'(I_r \times T'T \times \underset{\sim}{1}_n)(I_{rs} \times \underset{\sim}{1}_n')\underset{\sim}{y}$$

$$= n^{-2}\underset{\sim}{y}'(I_r \times T'T \times J_n)\underset{\sim}{y}$$

$$= n^{-2}\underset{\sim}{y}'[I_r \times (I_s - \frac{1}{s}J_s) \times J_n]\underset{\sim}{y}$$

$$= n^{-2}\underset{\sim}{y}'[(I_{rs} \times J_n) - \frac{1}{s}(I_r \times J_{sn})]\underset{\sim}{y}. \qquad (3.110)$$

Hence,

$$\underset{\sim}{w}_{t2}'\underset{\sim}{w}_{t2} = n^{-1}\underset{\sim}{y}'[n^{-1}(I_{rs} \times J_n) - \frac{1}{sn}(I_r \times J_{sn})]\underset{\sim}{y}$$

$$- n^{-1}\underset{\sim}{y}'P_2\underset{\sim}{y}$$

$$= n^{-1}\underset{\sim}{y}'[n^{-1}(I_{rs} \times J_n) - \frac{1}{sn}(I_r \times J_{sn}) - P_2]\underset{\sim}{y}$$

Note from (3.108) that

$$P_3 = n^{-1}(I_{rs} \times J_n) - \frac{1}{sn}(I_r \times J_{sn}) - P_2,$$

therefore,

$$\underset{\sim}{w}'_{t2}\underset{\sim}{w}_{t2} = n^{-1}\underset{\sim}{y}'P_3\underset{\sim}{y}.$$

Consequently,

$$\begin{aligned} F_{t1} &= \frac{\underset{\sim}{w}'_{t1}\underset{\sim}{w}_{t1}/(s-1)}{\underset{\sim}{w}'_{t2}\underset{\sim}{w}_{t2}/(r-1)(s-1)}, \\ &= \frac{n^{-1}(\underset{\sim}{y}'P_2\underset{\sim}{y})/(s-1)}{n^{-1}(\underset{\sim}{y}'P_3\underset{\sim}{y})/(r-1)(s-1)} \\ &= F_3 \text{ .} \qquad \square \end{aligned}$$

B. Reduction of the Exact Test for the Fixed Effects

From (3.93)

$$\underset{\sim}{w}^* = \underset{\sim}{\mu}^*_{2:4} + (\lambda^F_{max} I_{\nu_1} - L_{2:4})H'_{F1}\underset{\sim}{y},$$

where, $(\nu_1 = s(r-1))$;

$$L_{2:4} = \begin{bmatrix} P_2^{*}KP_2^{*\prime} & P_2^{*}KP_4^{*\prime} \\ P_4^{*}KP_2^{*\prime} & P_4^{*}KP_4^{*\prime} \end{bmatrix} ;$$

$$\lambda_{\max}^{F} = \text{largest eigenvalue of } L_{2:4}.$$

When we have a balanced design the matrix K reduces to

$$K = \bigoplus_{i,j}^{r,s} n_{ij}^{-1} = \bigoplus_{i,j}^{r,s} n^{-1} = n^{-1} I_{rs}. \qquad (3.111)$$

Hence, the matrix, $L_{2:4}$, reduces to

$$L_{2:4} = n^{-1} \begin{bmatrix} P_2^{*}P_2^{*\prime} & P_2^{*}P_4^{*\prime} \\ P_4^{*}P_2^{*\prime} & P_4^{*}P_4^{*\prime} \end{bmatrix} \qquad (3.112)$$

$$= n^{-1} I_{s(r-1)} \qquad \left(\text{see } (3.88)\right).$$

Consequently, $\lambda_{\max}^{F} = n^{-1}$, and therefore when we have a balanced design, then $\underset{\sim}{w}^{*}$ becomes

$$\underset{\sim}{w}^{*} = \underset{\sim}{\mu}_{2:4} \qquad (3.113)$$

$$= P_{2:4}^{*} B_1 \underset{\sim}{\alpha} + P_{2:4}^{*} \underset{\sim}{(\alpha\beta)} + P_{2:4}^{*} \bar{\underset{\sim}{e}}.$$

Proceeding as in (3.97) we again have $X^{*} = P_{2:4}^{*} B_1$, but now

$$\underset{\sim}{\varepsilon}_b^* = P_{2:4}^*\left[(\underset{\sim}{\alpha\beta}) + \bar{\underset{\sim}{e}}\right], \qquad (3.114)$$

(Note that $\mathrm{Var}(\underset{\sim}{\varepsilon}^*) = I(\underset{\sim}{\sigma}_{\alpha\beta} + n^{-1}\sigma_e^2)$) and we can express (3.113) as

$$\underset{\sim}{w}^* = X^*\underset{\sim}{\alpha} + \underset{\sim}{\varepsilon}^* \ . \qquad (3.115)$$

Recall from (3.106) that

$$F^*(H) = \frac{\underset{\sim}{\ell}_a^{*\prime}\left(R^{*\prime} G_{X^*} R^*\right)^{-1} \underset{\sim}{\ell}_a^*/g}{SSE^*/(r-1)(s-1)},$$

was the test statistic derived to test the H_o: $R^{*\prime}\underset{\sim}{\alpha} = \underset{\sim}{m}^*$. We also recall that $\underset{\sim}{\ell}_a^*$ from (3.102) and SSE^* from (3.105) are

$$\underset{\sim}{\ell}_a^* = R^{*\prime}\underset{\sim}{\alpha}^o - \underset{\sim}{m}^*$$

and

$$SSE^* = \underset{\sim}{w}^{*\prime}\left(I_{s(r-1)} - X^*(X^{*\prime}X^*)^{-1}X^{*\prime}\right)\underset{\sim}{w}^* \ .$$

Lemma 3.10

When we have a balanced design

$$F^*(H) = \frac{MS\alpha}{MS\alpha\beta},$$

when $g = r-1$ and $\underset{\sim}{m}^{*} = 0$ (see (3.106)), that is, when the hypothesis to be tested is $R^{*\prime}\underset{\sim}{\alpha} = \underset{\sim}{0}$, and g equals to the maximum number of linearly independent estimable functions of $\underset{\sim}{\alpha}$.

Proof: When we have a balanced design

$$SSE^{*} = \underset{\sim}{\mu}^{*\prime}_{2:4}\left(I_{s(r-1)} - X^{*}(X^{*\prime}X^{*})^{-}X^{*\prime}\right)\underset{\sim}{\mu}^{*}_{2:4}$$

Now, $X^{*\prime}X^{*} = B_1^{\prime}P^{*\prime}_{2:4}P^{*}_{2:4}B_1$. But, $P^{*\prime}_{2:4}P^{*}_{2:4} = I_{rs} - P^{*\prime}_{1:3}P^{*}_{1:3}$; hence,

$$B_1^{\prime}P^{*\prime}_{2:4}P^{*}_{2:4}B_1 = B_1^{\prime}B_1 - B_1^{\prime}P^{*\prime}_{1:3}P^{*}_{1:3}B_1$$

$$= (I_r \times \underset{\sim}{1}_s^{\prime})(I_r \times \underset{\sim}{1}_s) - B_1^{\prime}\left(\frac{1}{\sqrt{rs}}\underset{\sim}{1}_{rs} : P_3^{*\prime}\right)\begin{bmatrix} \frac{1}{\sqrt{rs}}\underset{\sim}{1}_{rs}^{\prime} \\ P_3^{*} \end{bmatrix} B_1$$

$$= sI_r - B_1^{\prime}\left(\frac{1}{rs}J_{rs} + P_3^{*\prime}P_3^{*}\right)B_1 .$$

Recall that P_3^{*} consists of the $s-1$ rows of P^{*} such that $P_3^{*}A_1P_3^{*\prime} = \underset{\sim}{0}_{(s-1)\times(s-1)}$. Hence, $P_3^{*}B_1B_1^{\prime}P_3^{*\prime} = \underset{\sim}{0}$, that is, $P_3^{*}B_1 = \underset{\sim}{0}$.

Consequently, $B_1^{\prime}P^{*\prime}_{2:4}P^{*}_{2:4}B_1 = sI_r - B_1^{\prime}\left(\frac{1}{rs}J_{rs}\right)B_1$

$$= sI_r - \frac{1}{rs}\left(s^2 J_r\right)$$

$$= sI_r - \frac{s}{r}J_r$$

$$= s\left(I_r - \frac{1}{r}J_r\right) .$$

Note that $\left(I_r - \frac{1}{r}J_r\right)\left(I_r - \frac{1}{r}J_r\right) = I_r - \frac{1}{r}J_r$, that is, $I_r - \frac{1}{r}J_r$ is an idempotent matrix. From Graybill (1983, p.111, Theorem 6.2.14), $\left(I_r - \frac{1}{r}J_r\right)^-$ equals $\left(I_r - \frac{1}{r}J_r\right)$. Hence,

$$(X^{*\prime}X^*)^- = \left[s\left(I_r - \frac{1}{r}J_r\right)\right]^- = \frac{1}{s}\left(I_r - \frac{1}{r}J_r\right). \qquad (3.116)$$

Now,

$$SSE^* = \underset{\sim}{\bar{y}}P^{*\prime}_{2:4}\left(I_{s(r-1)} - X^*(X^{*\prime}X^*)^-X^{*\prime}\right)P^*_{2:4}\underset{\sim}{\bar{y}}$$

$$= \underset{\sim}{\bar{y}}'\left(P^{*\prime}_{2:4}P^*_{2:4} - P^{*\prime}_{2:4}X^*(X^{*\prime}X^*)^-X^{*\prime}P^*_{2:4}\right)\underset{\sim}{\bar{y}},$$

where

$$P^{*\prime}_{2:4}X^* = P^{*\prime}_{2:4}P^*_{2:4}B_1 = \left(I_{rs} - P^{*\prime}_{1:3}P^*_{1:3}\right)B_1$$

$$= B_1 - \frac{1}{rs}J_{rs}B_1$$

$$= \left(I_r \times \underset{\sim}{1}_s\right) - \frac{1}{r}\left(J_r \times \underset{\sim}{1}_s\right).$$

Hence,

$$P^{*\prime}_{2:4}X^{*}(X^{*\prime}X^{*})^{-}X^{*\prime}P^{*}_{2:4}$$

$$= [(I_r\times\underset{\sim}{1}_s) - \frac{1}{r}(J_r\times\underset{\sim}{1}_s)][\frac{1}{s}(I_r - \frac{1}{r}J_r)]X^{*\prime}P^{*}_{2:4}$$

$$= \{[\frac{1}{s}(I_r - \frac{1}{r}J_r)\times\underset{\sim}{1}_s]-[\frac{1}{rs}(J_r-J_r)\times\underset{\sim}{1}_s]\}X^{*\prime}P^{*}_{2:4}$$

$$= \{\frac{1}{s}(I_r - \frac{1}{r}J_r)\times\underset{\sim}{1}_s\}X^{*\prime}P^{*}_{2:4}$$

$$= \{\frac{1}{s}(I_r - \frac{1}{r}J_r)\times\underset{\sim}{1}_s\}\{(I_r\times\underset{\sim}{1}^{\prime}_s) - \frac{1}{r}(J_r\times\underset{\sim}{1}^{\prime}_s)\}$$

$$= \frac{1}{s}(I_r - \frac{1}{r}J_r)\times J_s - \underset{\sim}{0}_{rs\times rs}$$

$$= \frac{1}{s}(I_r - \frac{1}{r}J_r)\times J_s. \qquad (3.117)$$

We have

$$SSE^{*} = \bar{\underset{\sim}{y}}^{\prime}(P^{*\prime}_{2:4}P^{*}_{2:4} - \frac{1}{s}(I_r - \frac{1}{r}J_r)\times J_s)\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}^{\prime}P^{*\prime}_{2:4}P^{*}_{2:4}\bar{\underset{\sim}{y}} - \frac{1}{s}\bar{\underset{\sim}{y}}^{\prime}[(I_r - \frac{1}{r}J_r)\times J_s]\bar{\underset{\sim}{y}} \quad .$$

Now,

$$\underset{\sim}{w}^{*\prime}(X^{*}(X^{*\prime}X^{*})^{-}X^{*\prime})\underset{\sim}{w}^{*}$$

$$= \bar{\underset{\sim}{y}}^{\prime}P^{\prime}_{2:4}X^{*}(X^{*\prime}X^{*})^{-}X^{*\prime}P_{2:4}\bar{\underset{\sim}{y}}$$

$$= \frac{1}{s}\,\bar{\underset{\sim}{y}}'((I_r - \frac{1}{r}J_r)\times J_s)\bar{\underset{\sim}{y}} \qquad \text{(see (3.117))}$$

$$= \frac{1}{sn^2}\,\underset{\sim}{y}'\,[(I_{rs}\times \underset{\sim}{1}_n)\{(I_r\times J_s) - \frac{1}{r}J_{rs}\}(I_{rs}\times \underset{\sim}{1}_n')]\underset{\sim}{y}$$

$$= \frac{1}{n}\,\underset{\sim}{y}'[\frac{1}{sn}(I_r\times J_{sn}) - \frac{1}{rsn}J_{rsn}]\underset{\sim}{y}$$

$$= \frac{1}{n}\,SS\alpha \quad \text{(see Khuri (1982))}. \qquad (3.118)$$

But,

$$\underset{\sim}{w}^{*\prime}(X^*(X^{*\prime}X^*)^-X^{*\prime})\underset{\sim}{w}^*$$

$$= \bar{\underset{\sim}{y}}'(P_2^{*\prime}P_2^* + P_4^{*\prime}P_4^*)B_1\,(X^{*\prime}X^*)^-B_1'(P_2^{*\prime}P_2^* + P_4^{*\prime}P_4^*)\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'[P_2^{*\prime}P_2^*B_1(X^{*\prime}X^*)^-B_1'P_2^{*\prime}P_2^*]\bar{\underset{\sim}{y}}$$

$$+\ \bar{\underset{\sim}{y}}'[P_4^{*\prime}P_4^*B_1(X^{*\prime}X^*)^-B_1'P_4^{*\prime}P_4^*]\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'[P_2^{*\prime}P_2^*B_1(X^{*\prime}X^*)^-B_1'P_2^{*\prime}P_2^*]\bar{\underset{\sim}{y}} \qquad (\text{since } P_4^*B_1 = \underset{\sim}{0}).$$

$$= \frac{1}{s}\,\bar{\underset{\sim}{y}}'[P_2^{*\prime}P_2^*(I_r\times\underset{\sim}{1}_s)(I_r - \frac{1}{r}J_r)(I_r\times\underset{\sim}{1}_s')P_2^{*\prime}P_2^*]\bar{\underset{\sim}{y}} \quad \text{(see(3.116))}$$

$$= \frac{1}{s}\,\bar{\underset{\sim}{y}}'[P_2^{*\prime}P_2^*\{(I_r\times J_s) - \frac{1}{r}(J_r\times J_s)\}]P_2^{*\prime}P_2^*\bar{\underset{\sim}{y}}$$

$$= \frac{1}{s}\, \bar{\underset{\sim}{y}}' \overset{*}{P}_2{}' \left[\{ \overset{*}{P}_2 (I_r \times J_s) \overset{*}{P}_2{}' \} - \frac{1}{r} \{ \overset{*}{P}_2 (J_r \times J_s) \overset{*}{P}_2{}' \} \right] \overset{*}{P}_2 \bar{\underset{\sim}{y}}$$

$$= \frac{1}{s}\, \bar{\underset{\sim}{y}}' \overset{*}{P}_2{}' \left[sI_{r-1} - \frac{1}{r} \{ \overset{*}{P}_2 (\underset{\sim}{1}_r \times I_s)(\underset{\sim}{1}_r' \times J_s) \overset{*}{P}_2{}' \} \right] \overset{*}{P}_2 \bar{\underset{\sim}{y}}$$

$$\left(\text{since, } I_r \times J_s = A_1 \text{ and } \overset{*}{P}_2 A_1 \overset{*}{P}_2{}' = sI_{r-1}\right)$$

$$= \bar{\underset{\sim}{y}}' \overset{*}{P}_2{}' \overset{*}{P}_2 \bar{\underset{\sim}{y}} \qquad \left(\underset{\sim}{1}_r \times I_s = B_1 \text{ and } \overset{*}{P}_2 B_1 = \underset{\sim}{0}\right).$$

Hence,

$$\bar{\underset{\sim}{y}}' \overset{*}{P}_2{}' \overset{*}{P}_2 \bar{\underset{\sim}{y}} = \frac{1}{n}\, SS\alpha. \qquad (3.119)$$

Now, recall that the r-1 rows of $\overset{*}{P}_2$ are the coefficients associated with orthogonal contrasts involving the levels of $\underset{\sim}{\alpha}$. Similarily, the (r-1)(s-1) rows of $\overset{*}{P}_4$ are the coefficients associated with orthogonal contrasts involving the levels of $\underset{\sim}{\alpha\beta}$. Analogously, we must have

$$\bar{\underset{\sim}{y}}' \overset{*}{P}_4{}' \overset{*}{P}_4 \bar{\underset{\sim}{y}} = \frac{1}{n}\, SS\alpha\beta. \qquad (3.120)$$

Furthermore,

$$\frac{1}{s}\, \bar{\underset{\sim}{y}}' \left[\left(I_r - \frac{1}{r} J_r \right) \times J_s \right] \bar{\underset{\sim}{y}}$$

$$= \frac{1}{s}\,\bar{\underset{\sim}{y}}'\left(I_r \times J_s - \frac{1}{r}J_{rs}\right)\bar{\underset{\sim}{y}}$$

$$= \frac{1}{sn^2}\,\underset{\sim}{y}'\left(I_{rs} \times \underset{\sim}{1}_n\right)\left(I_r \times J_s - \frac{1}{r}\,J_{rs}\right)\left(I_{rs} \times \underset{\sim}{1}_n'\right)\underset{\sim}{y}$$

$$= \frac{1}{n}\,\underset{\sim}{y}'\left[\frac{1}{sn}\left(I_r \times J_s \times \underset{\sim}{1}_n\right) - \frac{1}{rsn}\left(J_{rs} \times \underset{\sim}{1}_n\right)\right]\left(I_{rs} \times \underset{\sim}{1}_n'\right)\underset{\sim}{y}$$

$$= \frac{1}{n}\,\underset{\sim}{y}'\left[\frac{1}{sn}\left(I_r \times J_s \times J_n\right) - \frac{1}{rsn}\,J_{rsn}\right]\underset{\sim}{y}$$

$$= \frac{1}{n}\,SS\alpha \qquad \text{(see Khuri (1982)).} \qquad (3.121)$$

From (3.117), (3.118), (3.120), and (3.121) it follows that

$$SSE^* = \frac{1}{n}\left(SS\alpha + SS\alpha\beta - SS\alpha\right) = \frac{1}{n}\;SS\alpha\beta.$$

From Searle (1971, p. 199) and under the assumptions that $g = r-1$ and $\underset{\sim}{m}^* = \underset{\sim}{0}$,

$$\underset{\sim}{\ell}_a^{*\prime}\left(R^{*\prime}G_{X^*}R^*\right)^{-1}\underset{\sim}{\ell}_a^{*} = \underset{\sim}{w}^{*\prime}\left(X^*(X^{*\prime}X^*)^{-}X^{*\prime}\right)\underset{\sim}{w}^{*},$$

and from (3.118) we have

$$\underset{\sim}{\ell}_a^{*\prime}\left(R^{*\prime}G_{X^*}R^*\right)^{-1}\underset{\sim}{\ell}_a^{*} = \frac{1}{n}\,SS\alpha.$$

From the above results, it follows that when we have a balanced design and $g = r-1$ with $\underset{\sim}{m}^* = \underset{\sim}{0}$,

$$F^*(H) = \frac{\frac{1}{n}(SS\alpha)/g}{\frac{1}{n}(SS\alpha)/(r-1)(s-1)} = \frac{SS\alpha/(r-1)}{SS\alpha\beta/(r-1)(s-1)}$$

$$= \frac{MS\alpha}{MS\alpha\beta}. \qquad \square$$

3.5 The Design Parameter and the Power of the Test.

We will define $\lambda_{\max}$, $\lambda^{t}_{\max}$, and $\lambda^{F}_{\max}$ as the design parameters associated with the test statistics defined in (3.50), (3.84), and (3.106), respectively. The reason for using this terminology, "design parameter", will be clarified in the discussion that follows.

3.5.1 Design Parameter, $\lambda_{\max}$

Consider the test statistic F(H), defined in (3.50), to test the fixed effects in a 2-fold nested mixed model as pointed out in Section 3.3.2,

$$F(H) \sim F' \left[s, m_{\cdot} - a, (R'\underset{\sim}{\rho} - \underset{\sim}{m})'(R'G_X R)^{-1}(R'\underset{\sim}{\rho} - \underset{\sim}{m})/2\sigma^{*2} \right],$$

where $\sigma^{*2} = \sigma_b^2 + \lambda_{\max}\sigma_e^2$. Since the power of a test is an increasing function of the noncentrality parameter, the power of our test is a decreasing function of $\lambda_{\max}$. Note that the parameter $\lambda_{\max}$ is directly associated with the design, since $\lambda_{\max} = \max\{\lambda : |K-\lambda I| = 0\}$, where $K = \mathrm{Diag}\left(n^{-1}_{1(1)}, n^{-1}_{2(1)}, \ldots, n^{-1}_{m_a(a)}\right)$. Hence,

$$\lambda_{\max} = \max\left\{\lambda : \prod_{i,j} \left(n^{-1}_{j(i)} - \lambda\right) = 0\right\}$$

$$= \max\{\lambda : \lambda = n_{j(i)}^{-1} \text{ for } \forall_{ij}\}$$

$$= \max\left(n_{j(i)}^{-1}\right) = \left[\min\left(n_{j(i)}\right)\right]^{-1} \quad \forall_{ij} . \qquad (3.122)$$

Furthermore, we can find a lower bound for $\lambda_{\max}$.

Lemma 3.11

$$\frac{1}{m_{\bullet}} \sum_{i,j} n_{j(i)}^{-1} \leqslant \lambda_{\max}$$

(recall $m_{\bullet} = \sum_{i,j} m_i$).

Proof: $\left[\min\left(n_{j(i)}\right)\right]^{-1} \geqslant n_{j(i)}^{-1} \ \forall_{ij}$

$$\Rightarrow \sum_{i,j} \left[\min\left(n_{j(i)}\right)\right]^{-1} \geqslant \sum_{i,j} n_{j(i)}^{-1}$$

$$\Rightarrow \lambda_{\max} m_{\bullet} \geqslant \sum_{i,j} n_{j(i)}^{-1}$$

Hence, $\lambda_{\max} \geqslant \frac{1}{m_{\bullet}} \sum_{i,j} n_{j(i)}^{-1}$. $\square$

3.5.2 Design Parameter, λ^t_{max}:

Consider the test statistic F_{t1} defined in (3.84), for testing the hypothesis H_o: $\sigma^2_\beta = 0$, where σ^2_β is the variance component associated with the random main effect in the two-way cross classification model. The power associated with this test can be expressed as

$$P\left[F_{s-1,(r-1)(s-1)} > \theta\ F_{\alpha,s-1,(r-1)(s-1)}\right] \qquad \text{(see 4.31)}$$

where $\theta = \left\{1 + \left[r\sigma^2_b/\left(\sigma^2_{\alpha b} + \lambda^t_{max}\sigma^2_e\right)\right]\right\}^{-1}$. Hence, the power of F_{t1} is a decreasing function of λ^t_{max}, where λ^t_{max} is the largest eigenvalue of the matrix L_t and

$$L_t = PC_tKC_t'P'.$$

The matrix P is an orthogonal matrix whose rows are orthonormal eigenvectors of the matrix C_tKC_t'. The matrix C_t was defined as $I_r \times T$, where T is defined in (3.66), and

$K = \text{Diag}\left(n^{-1}_{11}, n^{-1}_{12}, \ldots, n^{-1}_{rs}\right)$.

Since the power of F_{t1} depends on λ^t_{max}, the following lemma is an important result.

Lemma 3.12

$$\frac{1}{r^2 s}\sum_{i,j} n_{ij}^{-1} \leq \lambda_{\max}^t \leq \frac{1}{\min\limits_{i,j}(n_{ij})}.$$

Proof: Let $e_{\max}^t(A)$ equal the largest eigenvalue of the matrix A. Since P is an orthogonal matrix then

$$e_{\max}^t\left[PC_t KC_t' P'\right] = e_{\max}^t C_t KC_t'.$$

Let $n_m = \min(n_{ij})\ \forall_{ij}$. We know that $C_t C_t' = I_r \times TT' = I_{r(s-1)}$. Hence, it follows that the matrix $n_m^{-1}\ C_t C_t' - C_t KC_t'$ is positive semidefinite and

therefore,

$$e_{\max}^t\left(n_m^{-1} C_t C_t'\right) \geq e_{\max}^t\left(C_t KC_t'\right) = \lambda_{\max}^t.$$

But, $e_{\max}^t\left(n_m^{-1} C_t C_t'\right) = e_{\max}^t\left(n_m^{-1} I_{r(s-1)}\right) = \frac{1}{n_m}$, that is,

$$\frac{1}{\min\limits_{i,j}(n_{ij})} \geq \lambda_{\max}^t\ . \tag{3.123}$$

Furthermore,

$$\frac{1}{r(s-1)}\,\mathrm{tr}\left(C_t KC_t'\right) = \frac{1}{r(s-1)}\sum_{i=1}^{r(s-1)} \lambda_i^t\ ,$$

where λ_i^t are the eigenvalues of $C_t KC_t'$. Hence, $\frac{1}{r(s-1)}\Sigma\lambda_i^t$ is the average of the eigenvalues of $C_t KC_t'$. Since $\lambda_{\max}^t \geq \frac{1}{r(s-1)}\Sigma\lambda_i^t$, then

$$\lambda^t_{\max} \geq \frac{1}{r(s-1)}\ \mathrm{tr}\left(C_t K C_t'\right). \tag{3.124}$$

Now,

$$\mathrm{tr}\left(C_t K C_t'\right) = \mathrm{tr}\left(C_t' C_t K\right)$$

$$= \mathrm{tr}\left\{\left(I_r \times \left[I_s - \frac{1}{s} J_s\right]\right) K\right\} \quad \text{(See appendix (D))}$$

$$= \mathrm{tr}\left\{I_{rs} K - \frac{1}{s}\left(I_r \times J_s\right) K\right\}$$

$$= \mathrm{tr}(K) - \frac{1}{s}\ \mathrm{tr}\left[\left(I_r \times J_s\right) K\right]$$

$$= \sum_{i,j} \frac{1}{n_{ij}} - \frac{1}{s} \sum_{i,j} \frac{1}{n_{ij}}\ .$$

From (3.122)

$$\lambda^t_{\max} \geq \frac{1}{r(s-1)}\left[\sum_{i,j} n_{ij}^{-1} - \frac{1}{s} \sum_{i,j} n_{ij}^{-1}\right]$$

$$= \frac{1}{r(s-1)}\ \frac{(s-1)}{s} \sum_{ij} n_{ij}^{-1}$$

$$= \frac{1}{rs} \sum_{ij} n_{ij}^{-1}\ . \tag{3.125}$$

The proof of lemma 3.12 follows from (3.123) and (3.125) □

3.5.3 Design Parameter, λ^F_{max}

The power of the test statistic used for testing the fixed effects, $F^*(H)$, also depends on a design parameter. To see this, recall from (3.106) that the parameter λ^F_{max} is equal to the largest eigenvalue of the matrix $L_{2:4} = P^*_{2:4} K P^{*\prime}_{2:4}$. Hence, the following lemma is an important result when we consider the power of this test.

Lemma 3.13

$$\frac{1}{r-1}\left[\frac{1}{s}\sum_{i,j}\frac{1}{n_{ij}} - \frac{1}{\min\limits_{i,j}(n_{ij})}\right] \leqslant \lambda^F_{max} \leqslant \frac{1}{\min\limits_{i,j}(n_{ij})} .$$

Proof: Let $e^F_{max}(A)$ be the largest eigenvalue of the maxtrix A and $n_m = \min(n_{ij}) \ \forall\, ij$. Then $P^*_{2:4}P^{*\prime}_{2:4} = I_{s(r-1)}$; hence, $n_m^{-1}P^*_{2:4}P^{*\prime}_{2:4} - P^*_{2:4}KP^{*\prime}_{2:4}$ is a positive semidefinite matrix, and therefore, $e^F_{max}\left(n_m^{-1}P^*_{2:4}P^{*\prime}_{2:4}\right) \geqslant e^F_{max}\left(P^*_{2:4}KP^{*\prime}_{2:4}\right)$. But,

$$e^F_{max}\left(n_m^{-1}P^*_{2:4}P^{*\prime}_{2:4}\right) = e^F_{max}\left(n_m^{-1}I_{s(r-1)}\right)$$

$$= \frac{1}{\min\limits_{i,j}}(n_{ij}) .$$

Consequently,

$$\frac{1}{\min\limits_{i,j}}(n_{ij}) \geq \lambda^{F}_{\max} .$$

Now, let λ^F_i denote the ith eigenvalue of $P^{*}_{2:4}KP^{*\prime}_{2:4}$. It follows that

$$\lambda^{F}_{\max} \geq \frac{1}{s(r-1)}\sum_{i}\lambda^{F}_{i} = \frac{1}{s(r-1)}\mathrm{tr}\left(P^{*}_{2:4}KP^{*\prime}_{2:4}\right)$$

$$= \frac{1}{s(r-1)}\mathrm{tr}\left(P^{*\prime}_{2:4}P^{*}_{2:4}K\right)$$

$$= \frac{1}{s(r-1)}\mathrm{tr}\left[\left(I_{s(r-1)} - P^{*\prime}_{1:3}P^{*}_{1:3}\right)K\right]$$

$$= \frac{1}{s(r-1)}\mathrm{tr}\left[K - P^{*\prime}_{1:3}P^{*}_{1:3}K\right]$$

$$= \frac{1}{s(r-1)}\mathrm{tr}(K) - \frac{1}{s(r-1)}\mathrm{tr}\left(P^{*\prime}_{1:3}P^{*}_{1:3}K\right) .$$

Now, $\mathrm{tr}\left(P^{*\prime}_{1:3}P^{*}_{1:3}K\right) = \mathrm{tr}\left(P^{*}_{1:3}KP^{*\prime}_{1:3}\right)$ and $P^{*}_{1:3}P^{*\prime}_{1:3} = I_s$, hence,

$$n_m^{-1}P^{*}_{1:3}P^{*\prime}_{1:3} \geq P^{*}_{1:3}KP^{*\prime}_{1:3}$$

(that is, the difference defined as the left-hand side minus the right-hand side is a positive semidefinite matrix). Therefore,

$$sn_m^{-1} = \mathrm{tr}\left(n_m^{-1} P_{1:3}^* P_{1:3}^{*\prime}\right) \geq \mathrm{tr}\left(P_{1:3}^* K P_{1:3}^{*\prime}\right),$$

or,

$$-\mathrm{tr}\left(P_{1:3}^* K P_{1:3}^{*\prime}\right) \geq -sn_m^{-1}.$$

Consequently,

$$\lambda_{\max}^F \geq \frac{1}{s(r-1)}\mathrm{tr}(K) - \frac{1}{s(r-1)}\mathrm{tr}\left(P_{1:3}^* K P_{1:3}^{*\prime}\right)$$

$$\geq \frac{1}{s(r-1)} \sum_{i,j} \frac{1}{n_{ij}} - \frac{1}{(r-1)} \frac{1}{\min\limits_{i,j}(n_{ij})} .$$

We have

$$\frac{1}{r-1}\left[\frac{1}{s} \sum_{i,j} \frac{1}{n_{ij}} - \frac{1}{\min\limits_{i,j}(n_{ij})}\right] \leq \lambda_{\max}^F \leq \frac{1}{\min\limits_{i,j}(n_{ij})} \qquad \square$$

CHAPTER FOUR
POWER STUDY

4.1 THE 2-FOLD NESTED MIXED MODEL POWER STUDY

4.1.1 Introduction

Consider again the unbalanced 2-fold nested mixed model in (3.21). The analysis of variance table is expressed as follows:

Table 4.1 Type I Analysis of Variance Table for the 2-Fold Nested Mixed Model

Source	Degrees of Freedom	Mean Squares	Expected Mean Squares
Nesting Effect ($\underset{\sim}{\alpha}$)	f_1	$MS(\underset{\sim}{\alpha}) = \underset{\sim}{y}' Q_1 \underset{\sim}{y}/f_1$	$Q(\underset{\sim}{\alpha})+\ell_1\sigma_b^2+\sigma_e^2$
Nested Effect ($\underset{\sim}{\beta}$)	f_2	$MS(\underset{\sim}{\beta}\|\underset{\sim}{\alpha}) = \underset{\sim}{y}' Q_2 \underset{\sim}{y}/f_2$	$\ell_2\sigma_b^2+\sigma_e^2$
Error	f_3	$MS_e = \underset{\sim}{y}' Q_3 \underset{\sim}{y}/f_3$	σ_e^2

where

$$f_1 = a-1,\ f_2 = m_{\bullet}-a, \text{ and } f_3 = n_{\bullet\bullet}-m_{\bullet};$$

$$Q_1 = \bigoplus_{i=1}^{a} \left(\frac{1}{n_{i.}} J_{n_{i.}}\right) - \frac{1}{n_{..}} J_{n_{..}},$$

$$Q_2 = \bigoplus_{i=1}^{a} \left\{ \bigoplus_{j=1}^{b_i} \frac{1}{n_{j(i)}} J_{n_{j(i)}} - \frac{1}{n_{i.}} J_{n_{i.}} \right\}, \text{ and}$$

$$Q_3 = I_{n_{..}} - \bigoplus_{i,j} \frac{1}{n_{j(i)}} J_{n_{j(i)}};$$

$$Q(\underset{\sim}{\alpha}) = [E(\underset{\sim}{y})]' Q_1 [E(\underset{\sim}{y})]/f_1$$

$$= \underset{\sim}{\alpha}' [\bigoplus_{i=1}^{a} (\underset{\sim}{1}'_{n_{i.}})] Q_1 [\bigoplus_{i=1}^{a} (\underset{\sim}{1}_{n_{i.}})] \underset{\sim}{\alpha}/f_1 .$$

Note that the above table is based on the type I sums of squares analysis in SAS (1985). When testing the nesting effect in a 2-fold nested model only the type I sum of squares are appropriate.

When we test for the significance of the nesting effect we are testing the hypothesis that $Q(\underset{\sim}{\alpha}) = 0$. But

$$Q(\underset{\sim}{\alpha}) = 0$$

$$\Leftrightarrow \underset{\sim}{\alpha}' [\bigoplus_{i=1}^{a} \underset{\sim}{1}'_{n_{i.}}] Q_1 [\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}}] \underset{\sim}{\alpha} = 0$$

$$\Leftrightarrow Q_1 [\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}}] \underset{\sim}{\alpha} = \underset{\sim}{0} \quad (Q_1 Q_1 = Q_1)$$

$$\Rightarrow \left[\bigoplus_{i=1}^{a} \left(\frac{1}{n_{..}} J_{n_{i.}}\right) - \frac{1}{n_{..}} J_{n_{..}} \right] \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} \right) \underset{\sim}{\alpha} = \underset{\sim}{0}$$

$$\Leftrightarrow \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i.}} \right) \underset{\sim}{\alpha} - \left(\frac{1}{n_{..}} \sum_{i=1}^{a} n_{i.} \alpha_i \right) \underset{\sim}{1}_{n_{..}} = \underset{\sim}{0} \quad . \qquad (4.1)$$

It can be shown that (4.1) implies $\alpha_1 = \alpha_2 = \cdots = \alpha_a$. Hence, when we test the significance of the fixed effects, we are testing the hypothesis H_o: $\alpha_1 = \alpha_2 = \cdots = \alpha_a$.

Note from Table 4.1, that under the hypothesis H_o, we cannot find two mean squares whose expectation values are the same. Hence, we need to use Satterthwaite's approximation to derive an approximate test for this hypothesis. This test is of the form

$$F' = \frac{\underset{\sim}{y}' Q_1 \underset{\sim}{y} / f_1}{\left\{(\ell_1/\ell_2) \frac{1}{f_2} \underset{\sim}{y}' Q_2 \underset{\sim}{y} - \left(\frac{\ell_1}{\ell_2} - 1\right) \frac{1}{f_3} \underset{\sim}{y}' Q_3 \underset{\sim}{y}\right\}} \quad . \qquad (4.2)$$

From Satterthwaite (1946), the number of denominator degrees of freedom is estimated by

$$v_2' = \frac{\left\{\frac{\ell_1}{\ell_2} MS(\underset{\sim}{\beta} | \underset{\sim}{\alpha}) - \left(\frac{\ell_1}{\ell_2} - 1\right) MS_e\right\}^2}{\frac{\left[\frac{\ell_1}{\ell_2} MS(\underset{\sim}{\beta} | \underset{\sim}{\alpha})\right]^2}{f_2} + \frac{\left[\left(\frac{\ell_1}{\ell_2} - 1\right) MS_e\right]^2}{f_3}} \quad . \qquad (4.3)$$

The F′ test (4.2), can be used to test H_o since the expectation value of the numerator and denominator are equivalent under this hypothesis, that is,

$$E_{H_o}\{\underset{\sim}{y}'Q_1\underset{\sim}{y}/f_1\} = \ell_1\sigma_b^2 + \sigma_e^2$$

$$= E\{(\frac{\ell_1}{\ell_2})\frac{1}{f_2}\underset{\sim}{y}'Q_2\underset{\sim}{y} - (\frac{\ell_1}{\ell_2} - 1)\frac{1}{f_3}\underset{\sim}{y}'Q_3\underset{\sim}{y}\},$$

where E_{H_o} denotes the expected value under H_o.

Recall that in CHAPTER THREE we derived an exact test for testing estimable linear functions of the fixed (nesting) effects, that is, for testing the hypothesis H_o: $R'\underset{\sim}{\alpha} = \underset{\sim}{m}$, where R' is a full row rank matrix. The test statistic

$$F(H) = \frac{\underset{\sim}{\ell}_a'(R'G_XR)^{-1}\underset{\sim}{\ell}_a/s}{SSE^{(M)}/(m_\bullet - a)},$$

has a central F with s and $(m_\bullet - a)$ numerator and denominator degrees of freedom, respectively, under H_o. Under H_a, F(H) has a noncentral F with noncentrality parameter,

$$n_c = \underset{\sim}{\ell}_a^{(M)'}(R'G_XR)^{-1}\underset{\sim}{\ell}_a^{(M)}/2(\sigma_b^2 + \lambda_{max}\sigma_e^2) \ . \qquad (4.4)$$

Since F(H) has an exact distribution, we can obtain the critical values of interest directly from the F table. On the other hand, the approximate test, F′, in (4.2) does not have a known distribution under H_o (or under H_a). Therefore, we have to use a Monte Carlo simulation to estimate the critical values for the approximate test statistic F´. Furthermore, in order to obtain the critical values for F´ it is necessary to specify σ_b^2 and σ_e^2, the variance components associated with the nested effect and error, respectively. It should also be noted that the exact test allows the testing of any group of independent estimable linear functions of the fixed effects, whereas the approximate test is designed for testing only the hypothesis H_o: $\alpha_1 = \alpha_2 = \cdots \alpha_a$.

In the next two sections we shall compare the power of the exact test with that of the approximate test F´.

4.1.2 Parameters and Designs Used in the Power Study

As previously mentioned, F´, does not have a known exact distribution under either the null or the alternative hypotheses. Hence, in order to study the power of F´ we need to estimate its critical values using a Monte Carlo simulation. Under H_o and with $\mu = 0$ and $\sigma_e^2 = 1$ (both without loss of generality), it follows from (3.21) that

$$E_{H_o}(\underset{\sim}{y}) = \underset{\sim}{0}$$

and (4.5)

$$\mathrm{Var}(\underset{\sim}{y}) = X_2X_2'\sigma_b^2 + I \ .$$

Making use of the RANNOR function in SAS (1985), we generated observations which are independently and normally distributed with mean zero and variance equal to unity. Let $\underset{\sim}{y}_1$ denote the vector of observations generated from the random number generator, RANNOR. Now consider the transformation, $\underset{\sim}{z}_1$,

$$\underset{\sim}{z}_1 = V_1'\underset{\sim}{y}_1, \tag{4.6}$$

Then, $\text{Var}(\underset{\sim}{z}_1) = V_1'V_1$, and V_1 is the Cholesky decomposition of the variance-covariance matrix of $\underset{\sim}{y}$. The random variable $\underset{\sim}{z}_1$ has the required expected vector and variance-covariance matrix to estimate the critical values of F'. That is,

$$E(\underset{\sim}{z}_1) = E_{H_o}(\underset{\sim}{y}) = \underset{\sim}{0} \tag{4.7}$$

and

$$\text{Var}(\underset{\sim}{z}_1) = \text{Var}(\underset{\sim}{y}) = X_2X_2'\sigma_b^2 + I \ .$$

For every vector $\underset{\sim}{z}_1$ generated, a value of F' is calculated. Ten samples consisting of 1000 F's each were generated for each design and values of variance components considered. For each sample the 90th, 95th and 99th percentiles of the calculated F's were obtained. The mean and its standard error for the ten F' quantiles were computed to estimate the critical values of interest.

Once estimates of the critical values of F' are obtained the associated power can be determined, not through further simulation, but by using Imhof's (1961) procedure. Essentially, any quadratic form, Q^*, of independently distributed normal random variables with mean μ_i and variance equal to unity can be expressed as

$$Q^* = \underset{\sim}{x}'A\underset{\sim}{x} = \sum_{i=1}^{\ell} \lambda_i z_i^2, \tag{4.8}$$

where $\lambda_i (i=1,2,\ldots,\ell)$ are the nonzero eigenvalues of A; z_i^2's are independently distributed non-central χ^2 distributed random variables with one degree of freedom and noncentrality parameter $\zeta_i^2 (z_i \sim N(\zeta_i, 1))$. For any f, $P(Q^* > f)$ is then calculated using the cumulative distribution function of Q^*, which is obtained in Imhof's Procedure by inverting the characteristic function of Q^*. Let f^* denote a critical value obtained from the Monte Carlo simulation. The power of F' can be expressed as

$$P_{H_a}(F' > f^*), \tag{4.9}$$

where the probability value in (4.9) is obtained under H_a.

Now in order to use Imhof's Procedure we need to express (4.9) as a probability value associated with a quadratic form of independently distributed normal random variables with mean ζ_i and variance equal to unity, that is, as in (4.8). Consider again

(4.1),

$$F' = \frac{\underset{\sim}{y}' Q_1 \underset{\sim}{y}/f_1}{[(\frac{\ell_1}{\ell_2}) \frac{1}{f_2} \underset{\sim}{y}' Q_2 \underset{\sim}{y} - (\frac{\ell_1}{\ell_2} - 1) \frac{1}{f_3} \underset{\sim}{y}' Q_3 \underset{\sim}{y}} ,$$

and let $c_1 = \ell_1/\ell_2$ and $c_2 = 1-c_1$. The power can be expressed as

$$P_{H_a}(F' > f^*) = P_{H_a}\{\underset{\sim}{y}'[\frac{Q_1}{f_1} - \frac{c_1 f^*}{f_2} Q_2 - \frac{c_2 f^*}{f_3} Q_3]\underset{\sim}{y} > 0\} \quad . \qquad (4.10)$$

Now under H_a, the vector of observations $\underset{\sim}{y}$ has

$$E_{H_a}(\underset{\sim}{y}) = X_1\underset{\sim}{\alpha}^* \qquad (\mu = 0)$$

and (4.11)

$$\mathrm{Var}_{H_a}(\underset{\sim}{y}) = X_2X_2'\sigma_b^2 + I \quad (\text{or } \mathrm{Var}(\underset{\sim}{y})),$$

where $\underset{\sim}{\alpha}^*$ is the vector of fixed effects under H_a.

Let $\Sigma_1 = \mathrm{Var}(\underset{\sim}{y})$. The matrix Σ_1 is symmetric; hence, there exists an orthogonal matrix P such that

$$P\Sigma_1 P' = \Lambda_1, \qquad (4.12)$$

where Λ_1 is a diagonal matrix whose diagonal elements are the eigenvalues of Σ_1. It follows that $\Sigma_1^{1/2}$ can be expressed as

$$\Sigma_1^{1/2} = P'\Lambda_1^{1/2}P. \qquad (4.13)$$

Consider the transformation

$$\underset{\sim}{u} = \Sigma_1^{-1/2}\underset{\sim}{y} \; . \qquad (4.14)$$

The expected value and variance-covariance of $\underset{\sim}{u}$ under H_a can be written as,

$$E(\underset{\sim}{u}) = \Sigma_1^{-1/2} X_1 \underset{\sim}{\alpha}^*$$

and (4.15)

$$\begin{aligned} Var(\underset{\sim}{u}) &= \Sigma_1^{-1/2} Var(\underset{\sim}{y}) \Sigma_1^{-1/2} \\ &= \Sigma_1^{-1/2} \Sigma_1 \Sigma_1^{-1/2} = I_{n_{..}} , \end{aligned}$$

respectively. As a result,

$$\begin{aligned} P_{H_a}\{F' > f^*\} &= P_{H_a}\{\underset{\sim}{y}'\Sigma_1^{-1/2}\Sigma_1^{1/2}[\frac{Q_1}{f_1} - \frac{c_1 f^*}{f_2}Q_2 - \frac{c_2 f^*}{f_3}Q_3]\Sigma_1^{1/2}\Sigma_1^{-1/2}\underset{\sim}{y} > 0\} \\ &= P_{H_a}\{\underset{\sim}{u}'A\underset{\sim}{u} > 0\}, \qquad (4.16) \end{aligned}$$

where $A = \Sigma_1^{1/2}[\frac{Q_1}{f_1} - \frac{c_1 f^*}{f_2}Q_2 - \frac{c_2 f^*}{f_3}Q_3]\Sigma_1^{1/2}$

and

$$\underset{\sim}{u} \sim N(\Sigma_1^{-1/2} X_1 \underset{\sim}{\alpha}^*, \; I_{n_{..}}).$$

By using the transformation in (4.14) we can use Imhof's Procedure in Appendix G to compute the power of the approximate test F'.

The noncentrality parameter, n_c, in (4.4) for the exact test F(H) depends on σ_b^2, the variance component associated with the nested effect; hence, the power of F(H) depends on this parameter. Likewise, the power of the approximate test F′ depends on σ_b^2. Note that from (4.6), the critical values associated with F′ depend on σ_b^2 since the observations generated from the Monte Carlo simulation required specifying values for this parameter. Consequently, in the study that follows we consider a range of values for σ_b^2, in particular,

$$\sigma_b^2 = .25,\ .75,\ 1,\ 3,\ 6,\ \text{and } 16\ . \tag{4.17}$$

The power depends also on the vector of fixed effects specified by the alternative hypothesis H_a. We considered four vectors $\underset{\sim}{\alpha}_1^a$, $\underset{\sim}{\alpha}_2^a$, $\underset{\sim}{\alpha}_3^a$, and $\underset{\sim}{\alpha}_4^a$ which are ordered according to their Euclidean norm values,

$$\underset{\sim}{\alpha}_1^a = \begin{bmatrix} 0.5 \\ 0.75 \\ 1 \\ 1.25 \end{bmatrix} \quad ; \quad \underset{\sim}{\alpha}_2^a = \begin{bmatrix} 0.5 \\ 1.5 \\ 3.0 \\ 4.5 \end{bmatrix} \quad ;$$

$$\underset{\sim}{\alpha}_3^a = \begin{bmatrix} 1.0 \\ 1.5 \\ 3.0 \\ 6.0 \end{bmatrix} \quad ; \quad \underset{\sim}{\alpha}_4^a = \begin{bmatrix} 2.0 \\ 5.0 \\ 7.0 \\ 8.0 \end{bmatrix} \quad . \tag{4.18}$$

The superscript signifies that we are under H_a.

Now, when we have a balanced design,

$$F_b = \frac{MS(\underset{\sim}{\alpha})}{MS(\underset{\sim}{\beta}|\underset{\sim}{\alpha})} = \frac{\underset{\sim}{y}'Q_1\underset{\sim}{y}/f_1}{\underset{\sim}{y}'Q_2\underset{\sim}{y}/f_2}, \qquad (4.19)$$

(see Table 4.1) has an exact F distribution under H_o and, therefore, F_b provides an exact test for H_o. In general, when we have an unbalanced design, $MS(\underset{\sim}{\alpha})$ and $MS(\underset{\sim}{\beta}|\underset{\sim}{\alpha})$ are not constant multiples of chi-square variates and are not independently distributed. In practice this is ignored and Satterthwaite's approximation is used to derive a test for H_o. This is inadequate since this approximation requires that $MS(\underset{\sim}{\alpha})$ and $MS(\underset{\sim}{\beta}|\underset{\sim}{\alpha})$ have chi-square type distributions and be independently distributed. As previously mentioned there are some conditions under which one or both of these requirements are satisfied.

Under last-stage uniformity, that is, when $n_{ij} = n \;\; \forall_{ij}$, $MS(\underset{\sim}{\alpha})$ and $MS(\underset{\sim}{\beta}|\underset{\sim}{\alpha})$ are chi-square type distributions and are independently distributed. Hence, the conditions are correct for applying the Satterthwaite approximation however, there is no need to use this approximation, since F_b in (4.19) provides an exact test.

The condition of partial balance, that is, when $n_{ij} = n_{ij'} \;\; i$ and $j \neq j'$, as pointed out in Tietjen (1974), is a sufficient condition for the independence of $MS(\underset{\sim}{\alpha})$ and $MS(\underset{\sim}{\beta}|\underset{\sim}{\alpha})$. Hence, under partial balance we at least satisfy one of the requirements necessary for applying Satterthwaite's approximation.

From above, it is apparent that the degree of imbalance affects the power of the approximate test (we know it affects the exact test because of λ_{max}). We considered this in the power study by looking at designs with varying degrees of imbalance.

Khuri (1986) proposed measures to quantify the degree of departure of the design from last-stage uniformity, partial balance, or complete balance. These are for last-stage uniformity,

$$\phi_1 = \frac{1}{1 + c_1^2} , \tag{4.20}$$

where $c_1^2 = \sum\limits_{ij} \frac{(n_{ij} - \bar{n}_{..})^2}{\bar{n}_{..}} / n_{..} \quad (\bar{n}_{..} = n_{..}/m_{.})$,

and for partial balance,

$$\phi_2 = \frac{1}{1 + c_2^2}, \tag{4.21}$$

where $c_2^2 = \sum\limits_{ij} \frac{(n_{ij} - n_{i.}/m_i)^2}{n_{i.}/m_i} / n_{..}$.

The measure for complete balance will depend on the variation in both the m_i's and the n_{ij}'s; hence, to measure departure from complete balance we use the quantity

$$\phi_3 = \frac{1}{1 + \left(c_3^2 + c_1^2\right)} , \tag{4.22}$$

where $c_3^2 = [\sum_{i=1}^{a} (m_i - \bar{m}_.)^2/\bar{m}_.]/m_.$

$\left(\bar{m}_. = m_./a,\ c_1^2\right.$ is defined in $\left.(4.20)\right)$

We have

$$0 \leq \phi_1, \phi_2, \phi_3 \leq 1 . \tag{4.23}$$

For i=1,2,3, as ϕ_i approaches 1, the degree of imbalance becomes less severe. Hence, in the power study, we considered designs which reflect different degrees of imbalance based on ϕ_1, ϕ_2, and ϕ_3. The designs and results of the study follow. Note that Designs 1 and 2 have four levels for the fixed effects, Designs 3, 4, and 5 have three levels and Designs 6, 7, and 8 have two levels.

4.1.3 Results

Table 4.2. Power values of the Approximate and Exact Test for the 2-Fold Nested Mixed Model
(ϕ_i for i=1,2,3 are the measures of departure from last stage uniformity, partial balance, and complete balance, respectively.)

DESIGN 1 $\phi_1 = \phi_2 = \phi_3 = .60$

N = 44 a = 4 $m_1=m_2=m_3=m_4=2$

$n_{11}=1$, $n_{12}=10$; $n_{21}=1$, $n_{22}=10$; $n_{31}=1$, $n_{32}=10$; $n_{41}=1$, $n_{42}=10$

$\underset{\sim}{g}_1^a = (0.5, .75, 1.0, 1.25)$

σ_b^2/σ_e^2	α = .01 Approx.	 Exact	α = .05 Approx.	 Exact	α = .10 Approx.	 Exact
.25	.328	.013	.364	.064	.467	.124
.75	.164	.012	.244	.060	.251	.117
1.00	.122	.012	.213	.058	.199	.115
3.00	.050	.011	.084	.054	.138	.108
6.00	.024	.010	.072	.052	.128	.104
16.00	.010	.010	.054	.051	.114	.102

DESIGN 1 $\phi_1 = \phi_2 = \phi_3 = .60$

N = 44 a = 4 $m_1=m_2=m_3=m_4=2$

$n_{11}=1$, $n_{12}=10$; $n_{21}=1$, $n_{22}=10$; $n_{31}=1$, $n_{32}=10$; $n_{41}=1$, $n_{42}=10$

$\underset{\sim}{g}_2^a = (0.5, 1.5, 3.0, 4.5)$

σ_b^2/σ_e^2	α = .01 Approx.	 Exact	α = .05 Approx.	 Exact	α = .10 Approx.	 Exact
.25	.429	.164	.524	.498	.686	.696
.75	.205	.111	.444	.377	.513	.568
1.00	.071	.096	.276	.336	.316	.520
3.00	.068	.046	.095	.189	.192	.323
6.00	.028	.029	.090	.127	.129	.229
16.00	.013	.017	.062	.080	.121	.153

Table 4.2-continued

DESIGN 1 $\phi_1 = \phi_2 = \phi_3 = .60$

$N = 44$ $a = 4$ $m_1=m_2=m_3=m_4=2$

$n_{11}=1$, $n_{12}=10$; $n_{21}=1$, $n_{22}=10$; $n_{31}=1$, $n_{32}=10$; $n_{41}=1$, $n_{42}=10$

$\underset{\sim}{\alpha}_3^a = (1.0, 1.5, 3.0, 6.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.461	.295	.566	.710	.795	.874
.75	.126	.200	.446	.566	.624	.760
1.00	.085	.171	.324	.511	.516	.709
3.00	.072	.077	.097	.286	.202	.456
6.00	.027	.044	.094	.181	.134	.311
16.00	.015	.022	.065	.101	.126	.187

DESIGN 1 $\phi_1 = \phi_2 = \phi_3 = .60$

$N = 44$ $a = 4$ $m_1=m_2=m_3=m_4=2$

$n_{11}=1$, $n_{12}=10$; $n_{21}=1$, $n_{22}=10$; $n_{31}=1$, $n_{32}=10$; $n_{41}=1$, $n_{42}=10$

$\underset{\sim}{\alpha}_4^a = (2.0, 5.0, 7.0, 8.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.500	.418	.672	.838	.765	.949
.75	.264	.291	.433	.705	.524	.870
1.00	.121	.250	.211	.647	.328	.828
3.00	.088	.111	.144	.377	.325	.577
6.00	.043	.060	.117	.234	.124	.388
16.00	.015	.028	.089	.122	.100	.293

Table 4.2-continued

DESIGN 2 $\phi_1 = \phi_2 = \phi_3 = .70$

N = 44 a = 4 $m_1=m_2=m_3=m_4=2$

$n_{11}=1$, $n_{12}=10$; $n_{21}=1$, $n_{22}=10$; $n_{31}=3$, $n_{32}=8$; $n_{41}=3$, $n_{42}=8$

$\underset{\sim}{\alpha}_1^a = (0.5, .75, 1.0, 1.25)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.210	.013	.310	.064	.333	.124
.75	.086	.012	.118	.060	.174	.117
1.00	.043	.012	.104	.058	.126	.115
3.00	.024	.011	.083	.054	.119	.108
6.00	.015	.010	.070	.052	.118	.104
16.00	.011	.010	.051	.051	.091	.102

DESIGN 2 $\phi_1 = \phi_2 = \phi_3 = .70$

N = 44 a = 4 $m_1=m_2=m_3=m_4=2$

$n_{11}=1$, $n_{12}=10$; $n_{21}=1$, $n_{22}=10$; $n_{31}=3$, $n_{32}=8$; $n_{41}=3$, $n_{42}=8$

$\underset{\sim}{\alpha}_2^a = (0.5, 1.5, 3.0, 4.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.492	.164	.664	.498	.732	.696
.75	.173	.111	.298	.376	.312	.568
1.00	.038	.096	.278	.336	.301	.520
3.00	.021	.046	.097	.189	.134	.323
6.00	.014	.029	.084	.126	.114	.229
16.00	.014	.017	.049	.080	.090	.153

Table 4.2-continued

DESIGN 2 $\phi_1 = \phi_2 = \phi_3 = .70$

N = 44 a = 4 $m_1=m_2=m_3=m_4=2$

$n_{11}=1,\ n_{12}=10;\quad n_{21}=1,\ n_{22}=10;\quad n_{31}=3,\ n_{32}=8;\quad n_{41}=3,\ n_{42}=8$

$\underset{\sim}{\alpha}_3^a = (1.0,\ 1.5,\ 3.0,\ 6.0)$

	α = .01		α = .05		α = .10	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.444	.295	.761	.710	.801	.874
.75	.195	.200	.375	.566	.389	.760
1.00	.051	.171	.332	.511	.378	.709
3.00	.027	.077	.074	.286	.123	.456
6.00	.012	.044	.062	.181	.100	.311
16.00	.009	.022	.049	.101	.086	.187

DESIGN 2 $\phi_1 = \phi_2 = \phi_3 = .70$

N = 44 a = 4 $m_1=m_2=m_3=m_4=2$

$n_{11}=1,\ n_{12}=10;\quad n_{21}=1,\ n_{22}=10;\quad n_{31}=3,\ n_{32}=8;\quad n_{41}=3,\ n_{42}=8$

$\underset{\sim}{\alpha}_4^a = (2.0,\ 5.0,\ 7.0,\ 8.0)$

	α = .01		α = .05		α = .10	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.500	.418	.514	.838	.543	.949
.75	.103	.291	.218	.705	.372	.870
1.00	.074	.250	.156	.647	.222	.828
3.00	.052	.111	.061	.377	.168	.568
6.00	.023	.060	.045	.234	.106	.388
16.00	.010	.028	.032	.122	.099	.201

Table 4.2-continued

DESIGN 3 $\phi_1 = .63$ $\phi_2 = .59$ $\phi_3 = .57$

$N = 44$ $a = 3$ $m_1 = 2;\ m_2 = 3;\ m_3 = 4$

$n_{11}=4,\ n_{12}=11;\ n_{21}=1,\ n_{22}=2,\ n_{23}=12;\ n_{31}=1,\ n_{32}=5,\ n_{33}=7,\ n_{34}=1$

$\underset{\sim}{\sigma}_1^a = (0.5,\ .75,\ 1.00)$

σ_b^2/σ_e^2	$\alpha = .01$ Approx.	$\alpha = .01$ Exact	$\alpha = .05$ Approx.	$\alpha = .05$ Exact	$\alpha = .10$ Approx.	$\alpha = .10$ Exact
.25	.073	.013	.140	.063	.156	.123
.75	.041	.012	.126	.060	.145	.116
1.00	.033	.012	.057	.058	.142	.114
3.00	.022	.011	.048	.054	.106	.107
6.00	.016	.010	.046	.052	.103	.104
16.00	.016	.010	.046	.051	.102	.103

DESIGN 3 $\phi_1 = .63$ $\phi_2 = .59$ $\phi_3 = .57$

$N = 44$ $a = 3$ $m_1 = 2;\ m_2 = 3;\ m_3 = 4$

$n_{11}=4,\ n_{12}=11;\ n_{21}=1,\ n_{22}=2,\ n_{23}=12;\ n_{31}=1,\ n_{32}=5,\ n_{33}=7,\ n_{34}=1$

$\underset{\sim}{\sigma}_2^a = (0.5,\ 1.5,\ 3.0)$

σ_b^2/σ_e^2	$\alpha = .01$ Approx.	$\alpha = .01$ Exact	$\alpha = .05$ Approx.	$\alpha = .05$ Exact	$\alpha = .10$ Approx.	$\alpha = .10$ Exact
.25	.076	.168	.167	.454	.241	.626
.75	.047	.112	.124	.341	.179	.502
1.00	.023	.095	.035	.303	.176	.458
3.00	.019	.047	.025	.172	.067	.287
6.00	.015	.028	.019	.117	.054	.207
16.00	.010	.017	.013	.077	.032	.144

Table 4.2-continued

DESIGN 3 $\phi_1 = .63$ $\phi_2 = .59$ $\phi_3 = .57$

$N = 44$ $a = 3$ $m_1 = 2$; $m_2 = 3$; $m_3 = 4$

$n_{11}=4$, $n_{12}=11$; $n_{21}=1$, $n_{22}=2$, $n_{23}=12$; $n_{31}=1$, $n_{32}=5$, $n_{33}=7$, $n_{34}=1$

$\underset{\sim}{\alpha}_4^a = (2.0, 5.0, 7.0)$

σ_b^2/σ_e^2	$\alpha = .01$ Approx.	Exact	$\alpha = .05$ Approx.	Exact	$\alpha = .10$ Approx.	Exact
.25	.496	.693	.524	.947	.610	.986
.75	.132	.514	.283	.855	.312	.943
1.00	.089	.448	.103	.805	.161	.914
3.00	.058	.197	.063	.505	.124	.676
6.00	.019	.100	.046	.314	.110	.407
16.00	.008	.040	.036	.153	.106	.236

Table 4.2-continued

DESIGN 4 $\phi_1 = .76$ $\phi_2 = .74$ $\phi_3 = .70$

$N = 44$ $a = 3$ $m_1 = 2;\ m_2 = 3;\ m_3 = 4$

$n_{11}=4,\ n_{12}=8;\ n_{21}=1,\ n_{22}=5,\ n_{23}=10;\ n_{31}=4,\ n_{32}=4,\ n_{33}=4,\ n_{34}=4$

$\underset{\sim}{\alpha}_1^a = (0.5,\ .75,\ 1.00)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.033	.013	.088	.063	.243	.123
.75	.012	.012	.064	.060	.093	.116
1.00	.011	.012	.045	.058	.116	.114
3.00	.009	.011	.038	.054	.096	.107
6.00	.007	.011	.038	.052	.092	.104
16.00	.007	.010	.037	.051	.091	.102

DESIGN 4 $\phi_1 = .76$ $\phi_2 = .74$ $\phi_3 = .70$

$N = 44$ $a = 3$ $m_1 = 2;\ m_2 = 3;\ m_3 = 4$

$n_{11}=4,\ n_{12}=8;\ n_{21}=1,\ n_{22}=5,\ n_{23}=10;\ n_{31}=4,\ n_{32}=4,\ n_{33}=4,\ n_{34}=4$

$\underset{\sim}{\alpha}_2^a = (1.0,\ 1.5,\ 3.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.052	.168	.073	.454	.431	.626
.75	.021	.112	.076	.341	.184	.502
1.00	.011	.095	.043	.303	.078	.458
3.00	.010	.045	.024	.172	.052	.287
6.00	.009	.028	.018	.117	.032	.207
16.00	.008	.017	.016	.077	.027	.144

Table 4.2-continued

DESIGN 4 $\phi_1 = .76$ $\phi_2 = .74$ $\phi_3 = .70$

$N = 44$ $a = 3$ $m_1 = 2$; $m_2 = 3$; $m_3 = 4$

$n_{11}=4$, $n_{12}=8$; $n_{21}=1$, $n_{22}=5$, $n_{23}=10$; $n_{31}=4$, $n_{32}=4$, $n_{33}=4$, $n_{34}=4$

$\underset{\sim}{\alpha}_4^a = (2.0, 5.0, 7.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.498	.693	.532	.947	.555	.986
.75	.125	.514	.274	.855	.297	.943
1.00	.087	.448	.111	.805	.244	.914
3.00	.045	.197	.074	.505	.123	.676
6.00	.035	.099	.053	.314	.112	.471
16.00	.035	.040	.039	.153	.064	.236

Table 4.2-continued

DESIGN 5 $\phi_1 = .94$ $\phi_2 = .91$ $\phi_3 = .90$

$N = 44$ $a = 3$ $m_1 = m_2 = 3$; $m_3 = 4$

$n_{11}=5, n_{12}=4, n_{13}=3$; $n_{21}=3, n_{22}=5, n_{23}=8$; $n_{31}=4, n_{32}=4, n_{33}=4, n_{34}=4$

$\underset{\sim}{\alpha}_1^a = (0.5, .75, 1.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.025	.020	.096	.089	.183	.163
.75	.016	.015	.058	.070	.077	.134
1.00	.014	.014	.044	.066	.074	.127
3.00	.007	.012	.032	.056	.061	.111
6.00	.006	.011	.018	.053	.040	.105
16.00	.006	.010	.012	.051	.033	.102

DESIGN 5 $\phi_1 = .94$ $\phi_2 = .91$ $\phi_3 = .90$

$N = 44$ $a = 3$ $m_1 = m_2 = 3$; $m_3 = 4$

$n_{11}=5, n_{12}=4, n_{13}=3$; $n_{21}=3, n_{22}=5, n_{23}=8$; $n_{31}=4, n_{32}=4, n_{33}=4, n_{34}=4$

$\underset{\sim}{\alpha}_2^a = (1.0, 1.5, 3.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.031	.576	.042	.881	.112	.953
.75	.015	.290	.029	.624	.110	.777
1.00	.012	.224	.027	.533	.056	.696
3.00	.006	.073	.027	.243	.033	.379
6.00	.005	.038	.023	.147	.032	.250
16.00	.005	.020	.014	.086	.011	.158

Table 4.2-continued

DESIGN 5 $\phi_1 = .94$ $\phi_2 = .91$ $\phi_3 = .90$

$N = 44$ $a = 3$ $m_1 = m_2 = 3$; $m_3 = 4$

$n_{11}=5, n_{12}=4, n_{13}=3$; $n_{21}=3, n_{22}=5, n_{23}=8$; $n_{31}=4, n_{32}=4, n_{33}=4, n_{34}=4$

$\underset{\sim}{\alpha}{}_4^a = (2.0,\ 5.0,\ 7.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.497	.997	.550	.999	.643	1.00
.75	.224	.919	.312	.995	.555	.999
1.00	.113	.845	.221	.984	.288	.997
3.00	.095	.380	.106	.727	.176	.857
6.00	.034	.172	.087	.450	.100	.615
16.00	.006	.057	.022	.200	.065	.312

Table 4.2-continued

DESIGN 6 $\phi_1 = .49 \quad \phi_2 = \phi_3 = .43$

$N = 44 \quad a = 2 \quad m_1 = m_2 = 2$

$n_{11}=1,\ n_{12}=10;\quad n_{21}=1,\ n_{22}=32$

$\underset{\sim}{\alpha}_1^a = (0.5,\ .75)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.465	.010	.538	.052	.559	.104
.75	.306	.010	.344	.052	.406	.103
1.00	.281	.010	.315	.051	.357	.103
3.00	.134	.010	.185	.051	.236	.101
6.00	.084	.010	.128	.050	.166	.101
16.00	.045	.010	.077	.050	.135	.100

DESIGN 6 $\phi_1 = .49 \quad \phi_2 = \phi_3 = .43$

$N = 44 \quad a = 2 \quad m_1 = m_2 = 2$

$n_{11}=1,\ n_{12}=10;\quad n_{21}=1,\ n_{22}=32$

$\underset{\sim}{\alpha}_3^a = (1.0,\ 1.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.478	.012	.579	.059	.559	.117
.75	.300	.011	.361	.057	.419	.112
1.00	.290	.011	.322	.056	.379	.110
3.00	.131	.011	.187	.053	.234	.105
6.00	.086	.010	.131	.052	.169	.103
16.00	.045	.010	.077	.051	.136	.101

Table 4.2-continued

DESIGN 6 $\phi_1 = .49$ $\phi_2 = \phi_3 = .43$

$N = 44$ $a = 2$ $m_1 = m_2 = 2$

$n_{11}=1,\ n_{12}=10;\ n_{21}=1,\ n_{22}=32$

$\underset{\sim}{\alpha}_2^a = (0.5, 1.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.505	.018	.581	.086	.623	.166
.75	.324	.016	.359	.076	.410	.148
1.00	.290	.015	.317	.073	.366	.142
3.00	.134	.012	.197	.061	.232	.121
6.00	.085	.011	.129	.057	.167	.112
16.00	.045	.010	.077	.053	.135	.105

DESIGN 6 $\phi_1 = .49$ $\phi_2 = \phi_3 = .43$

$N = 44$ $a = 2$ $m_1 = m_2 = 2$

$n_{11}=1,\ n_{12}=10;\ n_{21}=1,\ n_{22}=32$

$\underset{\sim}{\alpha}_4^a = (2.0, 5.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.633	.078	.700	.331	.823	.546
.75	.370	.060	.474	.261	.510	.448
1.00	.361	.053	.419	.237	.500	.413
3.00	.117	.032	.215	.149	.356	.273
6.00	.103	.022	.156	.108	.194	.203
16.00	.049	.015	.083	.074	.111	.096

Table 4.2-continued

DESIGN 7 $\phi_1 = .76$ $\phi_2 = .71$ $\phi_3 = .50$
$N = 44$ $a = 2$ $m_1 = 2$; $m_2 = 3$
$n_{11}=1$, $n_{12}=7$; $n_{21}=13$, $n_{22}=17$, $n_{23}=6$
$\underset{\sim}{\alpha}_1^a = (0.5, .75)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.016	.011	.057	.054	.104	.106
.75	.015	.011	.049	.053	.098	.104
1.00	.011	.010	.048	.052	.096	.104
3.00	.010	.010	.044	.051	.096	.102
6.00	.010	.010	.042	.051	.090	.101
16.00	.009	.010	.041	.050	.084	.100

DESIGN 7 $\phi_1 = .76$ $\phi_2 = .71$ $\phi_3 = .50$
$N = 44$ $a = 2$ $m_1 = 2$; $m_2 = 3$
$n_{11}=1$, $n_{12}=7$; $n_{21}=13$, $n_{22}=17$, $n_{23}=6$
$\underset{\sim}{\alpha}_3^a = (1.0, 1.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.014	.014	.038	.065	.113	.125
.75	.013	.012	.029	.061	.108	.118
1.00	.010	.012	.025	.059	.105	.116
3.00	.008	.011	.017	.055	.103	.108
6.00	.008	.011	.016	.053	.107	.104
16.00	.006	.010	.016	.051	.108	.102

Table 4.2-continued

DESIGN 7 $\phi_1 = .76$ $\phi_2 = .71$ $\phi_3 = .50$

$N = 44$ $a = 2$ $m_1 = 2;\ m_2 = 3$

$n_{11}=1,\ n_{12}=7;\ n_{21}=13,\ n_{22}=17,\ n_{23}=6$

$\underset{\sim}{\sigma}_2^a = (0.5,\ 1.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.019	.024	.053	.109	.137	.199
.75	.017	.020	.049	.092	.098	.171
1.00	.011	.019	.053	.087	.110	.163
3.00	.010	.014	.047	.068	.109	.132
6.00	.010	.012	.051	.061	.108	.118
16.00	.009	.011	.050	.054	.109	.107

DESIGN 7 $\phi_1 = .76$ $\phi_2 = .71$ $\phi_3 = .50$

$N = 44$ $a = 2$ $m_1 = 2;\ m_2 = 3$

$n_{11}=1,\ n_{12}=7;\ n_{21}=13,\ n_{22}=17,\ n_{23}=6$

$\underset{\sim}{\sigma}_4^a = (2.0,\ 5.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.091	.164	.112	.519	.245	.721
.75	.087	.116	.063	.405	.145	.601
1.00	.034	.102	.056	.365	.125	.556
3.00	.015	.053	.068	.214	.083	.359
6.00	.014	.033	.067	.145	.079	.255
16.00	.014	.019	.061	.089	.069	.137

Table 4.2-continued

DESIGN 8 $\phi_1 = .98$ $\phi_2 = \phi_3 = .85$

$N = 44$ $a = 2$ $m_1 = m_2 = 2$

$n_{11}=14$, $n_{12}=17$; $n_{21}=8$, $n_{22}=5$

$\underset{\sim}{\alpha}_1^a = (0.5, .75)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.016	.011	.101	.056	.128	.112
.75	.011	.011	.094	.053	.124	.106
1.00	.012	.010	.086	.052	.117	.104
3.00	.013	.010	.076	.051	.101	.102
6.00	.013	.010	.076	.050	.091	.101
16.00	.015	.010	.074	.050	.083	.100

DESIGN 8 $\phi_1 = .98$ $\phi_2 = \phi_3 = .85$

$N = 44$ $a = 2$ $m_1 = m_2 = 2$

$n_{11}=14$, $n_{12}=17$; $n_{21}=8$, $n_{22}=5$

$\underset{\sim}{\alpha}_3^a = (1.0, 1.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.034	.015	.104	.075	.131	.146
.75	.014	.013	.068	.062	.130	.122
1.00	.013	.012	.065	.060	.103	.118
3.00	.013	.011	.067	.054	.087	.107
6.00	.008	.010	.066	.052	.085	.103
16.00	.006	.010	.064	.051	.076	.101

Table 4.2-continued

DESIGN 8 $\phi_1 = .98$ $\phi_2 = \phi_3 = .85$

$N = 44$ $a = 2$ $m_1 = m_2 = 2$

$n_{11}=14$, $n_{12}=17$; $n_{21}=8$, $n_{22}=5$

$\underset{\sim}{\alpha}_2^a = (0.5, 1.5)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.088	.032	.124	.148	.165	.271
.75	.050	.020	.069	.098	.128	.186
1.00	.041	.018	.058	.088	.110	.168
3.00	.032	.013	.055	.064	.107	.126
6.00	.017	.012	.042	.057	.101	.114
16.00	.014	.011	.033	.053	.100	.105

DESIGN 8 $\phi_1 = .98$ $\phi_2 = \phi_3 = .85$

$N = 44$ $a = 2$ $m_1 = m_2 = 2$

$n_{11}=14$, $n_{12}=17$; $n_{21}=8$, $n_{22}=5$

$\underset{\sim}{\alpha}_4^a = (2.0, 5.0)$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$	
σ_b^2/σ_e^2	Approx.	Exact	Approx.	Exact	Approx.	Exact
.25	.114	.189	.123	.642	.214	.865
.75	.061	.099	.111	.402	.189	.634
1.00	.036	.081	.106	.341	.131	.558
3.00	.013	.037	.053	.172	.060	.311
6.00	.011	.024	.050	.115	.052	.216
16.00	.008	.015	.043	.075	.045	.155

4.1.4 Analysis of Power Study

Consider Designs 1 and 2 having four levels for the nesting effect. These designs differ in their degree of imbalance with Design 1 slightly more imbalanced than Design 2. In both designs we notice that the power is greater for the approximate test than the exact test when we consider the vector, $\underset{\sim}{\alpha}_1^a$, of fixed effects under H_a. For $\underset{\sim}{\alpha}_2^a$, the approximate test continues to beat the exact test, but, only for the smaller values of $\Delta_b = \sigma_b^2/\sigma_e^2$ and, in general, at the 99th percentile. Note that when $\Delta_b \geqslant 1.0$ the exact test soundly defeats the approximate test. This continues to be more pronounced as we look at the tables for $\underset{\sim}{\alpha}_3^a$ and $\underset{\sim}{\alpha}_4^a$. When we consider these vectors of fixed effects, the exact test, in general, has a better performance.

Hence, it appears at this stage that small departures from the null hypothesis and small values of Δ_b tends to favor the approximate test. As we move away from the null, the power of the exact test dominates the power of the approximate test.

Consider Designs 3, 4, and 5, having a low, medium, and high degree of imbalance, respectively. Design 3 continues the same pattern observed above for Designs 1 and 2, that is, the approximate test exhibiting greater power than the exact test for small departures from the null (actually only for $\underset{\sim}{\alpha}_1^a$). As we consider $\underset{\sim}{\alpha}_2^a$ and $\underset{\sim}{\alpha}_3^a$ for this design we note the increasingly greater power of the exact test as we move away from the null, with the gap between the powers as we increase Δ_b.

When we consider Design 4, having moderate imbalance, we have the same results except that the powers of the approximate test are in general smaller than the powers of the approximate test in Design 3. We find that the exact test on almost every occasion has greater power than the approximate test. Analogous to Design 4, in Design 5 the power of the exact test, in general, is larger than the power of the approximate test.

Finally, consider Designs 6, 7, and 8 having two levels for the fixed effects. Unlike the other designs discussed thus far, Design 6 has the approximate test with a power that almost completely dominates the power of the exact test. The moderately imbalanced Design 7, exhibits a lower power for the approximate test when compared to Design 6, and in general for this design the exact and approximate powers are close. Only, for $\underset{\sim}{\alpha}_4^a$ and the 95th and 99th percentiles is the power of the exact test clearly greater than the power of the approximate test. The results of Design 8 are very similar to the results of Design 7. Only with approximate and exact test having slightly greater power compared to the powers in Design 7.

Hence, we find that in general, the exact test has greater power than the approximate test as we move away from the null (except for Design 6). The power of the approximate test is particularly strong for designs having a greater degree of imbalance. The powers for both the approximate and exact test are a decreasing function of Δ_b. On several occasions this appears not to be the case for the approximate test, however, this is attributed to the variation that exists with the critical value estimates obtained through the Monte Carlo simulation.

4.2 Power Study for the Two-Way Cross Classification Mixed Model.

4.2.1 Introduction

Consider the two-way cross classification mixed model in (3.60). In the last chapter exact tests were developed for testing the significance of both, the variance component associated with the random main effect, and the fixed effects of the model. In this section we will consider a power comparison of the existing approximate procedures and the new derived tests.

Consider the analysis of variance tables of the unbalanced two-way cross classification with no missing cells (see Searle (1971, pp. 292-301)).

Table 4.3 Type I Analysis of Variance Table for the Two-Way Cross Classification Mixed Model

Source	Degrees of Freedom	Mean Squares	Expected Mean Squares
Main Effect ($\underset{\sim}{\alpha}$)	$r-1$	$MS(\underset{\sim}{\alpha} \mid \mu)$	$Q_1(\underset{\sim}{\alpha})+h_1\sigma^2_{\alpha\beta}+h_2\sigma^2_{\beta}+\sigma^2_e$
Main Effect ($\underset{\sim}{\beta}$)	$s-1$	$MS(\underset{\sim}{\beta} \mid \mu,\underset{\sim}{\alpha})$	$h_3\sigma^2_{\alpha\beta}+h_4\sigma^2_{\beta}+\sigma^2_e$
Interaction ($\underset{\sim}{\alpha\beta}$)	$(r-1)(s-1)$	$MS(\underset{\sim}{\alpha\beta} \mid \mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$	$h_5\sigma^2_{\alpha\beta}+\sigma^2_e$
Error	$n_{..}-rs$	MS_e	σ^2_e

where

$$MS(\underset{\sim}{\alpha}|\mu) = \sum_{i=1}^{r} \frac{y_{i..}^2/n_{i.} - y_{...}^2/n_{..}}{(r-1)} ,$$

$$MS(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha}) = \frac{\sum_{i=1}^{r} n_{i.}\bar{y}_{i..}^2 + \underset{\sim}{r}' G_1^{-1} \underset{\sim}{r} - \sum_{i=1}^{r} y_{i..}^2/n_{i.},}{(s-1)} \qquad (4.24)$$

$$MS(\underset{\sim}{\alpha\beta}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \frac{\sum_{i,j} y_{ij.}^2/n_{ij} - \sum_{i} n_{i.}\bar{y}_{i..}^2 - \underset{\sim}{r}' G_1^{-1} \underset{\sim}{r}}{(r-1)(s-1)} ,$$

$$MS_e = [\sum_{ijk} y_{ijk}^2 - \sum_{ij} n_{ij}\bar{y}_{ij.}^2]/(n_{..}-rs),$$

and

$$Q_1(\underset{\sim}{\alpha}) = [E(\underset{\sim}{y})]'\{\bigoplus_{i=1}^{r} \frac{1}{n_{i.}} J_{n_{i.}} - \frac{1}{n_{..}} J_{n_{..}}\}[E(\underset{\sim}{y})]/(r-1)$$

$$= \frac{\underset{\sim}{\alpha}'[\bigoplus_{i=1}^{r} \underset{\sim}{1}'_{n_{i.}}]\{\bigoplus_{i=1}^{r} \frac{1}{n_{i.}} J_{n_{i.}} - \frac{1}{n_{..}} J_{n_{..}}\}[\bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}}]\underset{\sim}{\alpha}}{(r-1)} ;$$

$$G_1 = \begin{cases} [c_{ij}] = n_{.j} = \sum\limits_{ij} n^2_{ij}/n_{i.} & \text{whenever} \quad i=j \leqslant s-1 \\ [c_{ij}] = -\sum\limits_{i=1}^{r} n_{ij}n'_{ij}/n_{i.} & \text{whenever} \quad i\neq j \leqslant s-1 \end{cases} \tag{4.25}$$

$$\underset{\sim}{r}_1 = (r_1, r_2, \ldots, r_{s-1})', \text{ where } r_j = \{y_{.j.} - \sum_i n_{ij}\bar{y}_{i..}\}$$

$$\text{for } j = 1,2,\ldots,s-1 \ .$$

Table 4.4 Type II Analysis of Variance Table for the Two-Way Cross Classification Mixed Model

Source	Degrees of Freedom	Mean Squares	Expected Mean Squares
Main Effect ($\underset{\sim}{\alpha}$)	$r-1$	$MS(\underset{\sim}{\alpha}\mid\mu,\underset{\sim}{\beta})$	$Q_2(\underset{\sim}{\alpha})+h_6\sigma^2_{\alpha\beta}+\sigma^2_e$
Main Effect ($\underset{\sim}{\beta}$)	$s-1$	$MS(\underset{\sim}{\beta}\mid\mu,\underset{\sim}{\alpha})$	$h_3\sigma^2_\beta+h_4\sigma^2_{\alpha\beta}+\sigma^2_e$
Interaction ($\underset{\sim}{\alpha\beta}$)	$(r-1)(s-1)$	$MS(\underset{\sim}{\alpha\beta}\mid\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$	$h_5\sigma^2_{\alpha\beta}+\sigma^2_e$
Error	$n_{..}-rs$	MS_e	σ^2_e

where

$$MS(\underset{\sim}{\alpha}|\mu,\underset{\sim}{\beta}) = \{\sum_i n_{i.}\bar{y}^2_{i..} + \underset{\sim}{r}'_1 G_1^{-1}\underset{\sim}{r}_1 - \sum_j n_{.j}\bar{y}^2_{.j.}\}/(r-1);$$

$$(4.26)$$

$$Q_2(\underset{\sim}{\alpha}) = \frac{[E(\underset{\sim}{y})]'\{X_{12}(X'_{12}X_{12})^- X'_{12} - \bigoplus_{i=1}^{r} \frac{1}{n_{i.}} J_{n_{i.}}\}[E(\underset{\sim}{y})]}{(r-1)}$$

$$= \frac{\underset{\sim}{\alpha}'[\bigoplus_{i=1}^{r} \underset{\sim}{1}'_{n_{i.}}]\{X_{12}(X'_{12}X_{12})^- X'_{12} - \bigoplus_{i=1}^{r} \frac{1}{n_{i.}} J_{n_{i.}}\}[\bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}}]\underset{\sim}{\alpha}}{(r-1)},$$

where $X_{12} = [\underset{\sim}{1}_{n_{..}} : \bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}} : X]$

$$(X = [\bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{1j}} : \bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{2j}} : \ldots \bigoplus_{j=1}^{s} \underset{\sim}{1}'_{n_{rj}}]').$$

Let $A_X = [\bigoplus_{i=1}^{r} \underset{\sim}{1}'_{n_{i.}}]\{X_{12}(X'_{12}X_{12})^- X'_{12} - \bigoplus_{i=1}^{r} \frac{1}{n_{i.}} J_{n_{i.}}\}[\bigoplus_{i=1}^{r} \underset{\sim}{1}_{n_{i.}}]$.

Consider

$$Q_2(\underset{\sim}{\alpha}) = 0 \iff \underset{\sim}{\alpha}'A_X\underset{\sim}{\alpha} = 0.$$

The matrix A_X is positive semidefinite, consequently, $A_X = A_X^{1/2}A_X^{1/2}$; hence,

$$\underset{\sim}{\alpha}'A_X\underset{\sim}{\alpha} = 0 \iff A_X^{1/2}\underset{\sim}{\alpha} = \underset{\sim}{0}, \quad \text{that is,}$$

$\underset{\sim}{a}$ is in the orthogonal complement of the column space of A_X. This space has dimension equal to r-1. Since $A_X\underset{\sim}{1}_r = \underset{\sim}{0}$, we must have $\underset{\sim}{\alpha} \parallel \underset{\sim}{1}_r$, therefore, the vector $\underset{\sim}{a}$ can be expressed as a scalar multiple of the vector $\underset{\sim}{1}_r$, that is, $\alpha_1 = \alpha_2 = \cdots \alpha_r$ when $Q(\underset{\sim}{\alpha}) = 0$. Hence, when we test the significance of the fixed effects using the type II sums of squares (Table 4.4), we are testing the hypothesis $H_o^{(1)}: \alpha_1 = \alpha_2 = \cdots = \alpha_r$.

In a fashion similar to the derivations in (4.1), one can show that testing the fixed effects using the type I sums of squares (Table 4.3) is equivalent to testing the above hypothesis $H_o^{(1)}: \alpha_1 = \alpha_2 = \cdots = \alpha_r$.

Using Tables 4.3 and 4.4 we can apply Satterthwaite's approximation and derive two approximate tests for the hypothesis $H_o^{(1)}$. Consider

$$F_{\underset{\sim}{\alpha}}^{(1)} = \frac{MS(\underset{\sim}{\alpha}|\mu)}{\rho_1}, \tag{4.27}$$

where

$$\rho_1 = \frac{h_2}{h_4} MS(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha}) + \frac{(h_1 - h_2h_3/h_4)}{h_5} MS(\alpha\beta|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$$
$$+ \left(1 - \left[h_2/h_4 + \frac{(h_1 - h_2h_3/h_4)}{h_5}\right]\right)MS_e \quad .$$

From Table 4.3,

$$E(\rho_1) = \frac{h_2}{h_4}\left\{h_3\sigma_{\alpha\beta}^2 + h_4\sigma_\beta^2 + \sigma_e^2\right\}$$

$$+ \frac{(h_1 - h_2h_3/h_4)}{h_5} \{h_5\sigma^2_{\alpha\beta} + \sigma^2_e\}$$

$$+ (1 - [\frac{h_2}{h_4} + \frac{h_1 - h_2h_3/h_4}{h_5}])\sigma^2_e$$

$$= E_{H_o^{(1)}}[MS\ (\underset{\sim}{\alpha}|\mu)].$$

Hence, $F^{(1)}_{\underset{\sim}{\alpha}}$ is an approximate test based on the type I sums of squares for testing the $H_o^{(1)}$: $\alpha_1 = \alpha_2 = \cdots = \alpha_r$.

In addition,

$$F^{(2)}_{\underset{\sim}{\alpha}} = \frac{MS(\underset{\sim}{\alpha}|\mu,\underset{\sim}{\beta})}{\rho^2}, \tag{4.28}$$

where

$$\rho_2 = \frac{h_6}{h_5} MS(\underset{\sim}{\alpha\beta}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) + (1 - \frac{h_6}{h_5})MS_e.$$

From Table 4.4,

$$E(\rho_2) = \frac{h_6}{h_5} \{h_9\sigma^2_{\alpha\beta} + \sigma^2_e\} + (1 - \frac{h_6}{h_5})\sigma^2_e$$

$$= E_{H_o^{(1)}}[MS(\underset{\sim}{\alpha}|\mu,\underset{\sim}{\beta})].$$

Hence, $F^{(2)}_{\underset{\sim}{\alpha}}$, based on the type II sums of squares, is also an approximate test for testing the fixed effects.

Finally, consider

$$F_{\underset{\sim}{\beta}} = \frac{MS(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha})}{\rho_3}, \qquad (4.29)$$

where

$$\rho_3 = \frac{h_4}{h_5} MS(\underset{\sim}{\alpha\beta}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) + \left(1 - \frac{h_4}{h_5}\right)MS_e$$

This test statistic can be used to test the hypothesis $H_o^{(2)}: \sigma_\beta^2 = 0$. Note that

$$E(\rho_3) = h_4\sigma_{\alpha\beta}^2 + \sigma_e^2 = E_{H_o^{(2)}}\left(MS(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha})\right).$$

These tests are approximate since the mean squares involved in forming the test statistics, in general, do not have chi-square type distributions and are not independently distributed. Furthermore, the denominator degrees of freedom of these F's are approximate. In the power study that follows we will compare the powers of $F_{\underset{\sim}{\alpha}}^{(1)}$ and $F_{\underset{\sim}{\alpha}}^{(2)}$ with $F^*(H)$ (see (3.106)) and $F_{\underset{\sim}{\beta}}$ with F_{t1} (see (3.84)), for testing the fixed effects and variance component associated with the random effects, respectively.

4.2.2 Parameters and Designs Used in the Power Study

Again, as with the approximate test derived for testing the fixed effects in an unbalanced two-fold nested mixed model, the approximate tests derived in the last section requires a Monte Carlo simulation to obtain the critical values of interest. The critical values are again obtained by producing ten samples of 1000 randomly generated observations of each of the three test statistics in (4.27), (4.28), and (4.29). The corresponding 90th, and 95th percentiles were then computed for each of the samples. The mean and the standard error of the quantiles obtained from the ten samples provide an estimate of the critical value.

Note that from (3.71), for a two-way cross classification model,

$$\mathrm{Var}(\underset{\sim}{y}) = X_2X_2'\sigma_\beta^2 + X_3X_3'\sigma_{\alpha\beta}^2 + I_{n_{\cdot\cdot}}\sigma_e^2 \ .$$

Hence, when obtaining the critical values and powers for $F_{\underset{\sim}{\alpha}}^{(1)}$ and $F_{\underset{\sim}{\beta}}^{(2)}$, the two approximate tests for testing the fixed effects, we had to specify values for σ_β^2 and $\sigma_{\alpha\beta}^2$. The critical values for the approximate test, $F_{\underset{\sim}{\beta}}$, for testing the hypothesis $H_o^{(2)}$: $\sigma_\beta^2 = 0$, requires specifying values only for $\sigma_{\alpha\beta}^2$. Note that under $H_o^{(2)}$, the variance-covariance of $\underset{\sim}{y}$ depends only on $\sigma_{\alpha\beta}^2$ and σ_e^2 (without loss of generality we set $\sigma_e^2 = 1$). We considered the following values for σ_β^2 and $\sigma_{\alpha\beta}^2$:

$$\begin{aligned} \sigma^2_\beta &= 0.5 \text{ and } 6.0 \\ \sigma^2_{\alpha\beta} &= 1.0 \text{ and } 4.0. \end{aligned} \qquad (4.30)$$

Having obtained the critical values, Imhof's Procedure in Appendix G (after transforming the vector $\underset{\sim}{y}$ to obtain the power of the appropriate quadratic form (see (4.8))) was used to obtain the approximate tests.

Now, recall that the exact test derived for testing the hypothesis $H_o^{(2)}$: $\sigma^2_\beta = 0$, is of the form

$$F_{t1} = \frac{\underset{\sim}{w}'_{t1}\underset{\sim}{w}_{t1}/(r-1)}{\underset{\sim}{w}'_{t2}\underset{\sim}{w}_{t2}/(r-1)(s-1)} ,$$

where $\underset{\sim}{w}_{t1}$ and $\underset{\sim}{w}_{t2}$ are described in (3.82) and are distributed as

$$\underset{\sim}{w}_{t1} \sim N_{(r-1)}[\underset{\sim}{0}, (r\sigma^2_\beta + \sigma^2_{\alpha\beta} + \lambda^t_{max}\sigma^2_e)]$$

and

$$\underset{\sim}{w}_{t2} \sim N_{(r-1(s-1)}[\underset{\sim}{0}, (\sigma^2_{\alpha\beta} + \lambda^t_{max}\sigma^2_e)].$$

Hence, under the hypothesis H_a: $\sigma^2_\beta \neq 0$,

$$F_{t1} \sim \frac{\sigma^2_{\alpha\beta} + \lambda^t_{max}\sigma^2_e}{r\sigma^2_\beta + \sigma^2_{\alpha\beta} + \lambda^t_{max}\sigma^2_e}^{\,-1} \cdot F_{\alpha,r-1,(r-1)(s-1)}, \qquad (4.31)$$

where $F_{\alpha,r-1,(r-1)(s-1)}$ is the upper α-percentile point of an F distribution with (r-1) and (r-1)(s-1) numerator and denominator degrees of freedom, respectively. Consequently, the power of F_{t1} for testing $H_o^{(2)}$, is given by

$$P \quad \frac{\sigma^2_{\alpha\beta}+\lambda^t_{\max}\sigma^2_e}{r\sigma^2_\beta+\sigma^2_{\alpha\beta}+\lambda^t_{\max}\sigma^2_e} \cdot F_{t1} > \frac{\sigma^2_{\alpha\beta}+\lambda^t_{\max}\sigma^2_e}{r\sigma^2_\beta+\sigma^2_{\alpha\beta}+\lambda^t_{\max}\sigma^2_e} \cdot F_{\alpha,r-1,(r-1)(s-1)}$$

$$= P\left[F_{r-1,(r-1)(s-1)} > \theta \cdot F_{\alpha,r-1,(r-1)(s-1)}\right], \qquad (4.32)$$

where $\theta = \left\{1 + r\sigma^2_\beta/\left(\sigma^2_{\alpha\beta}+\lambda^t_{\max}\sigma^2_e\right)\right\}^{-1}$.

As a result, the powers for F_{t1} were simply obtained by specifying values for σ^2_β and $\sigma^2_{\alpha\beta}$ (see (4.30)) and using the PROBF function in SAS (1985). The function PROBF computes the probability that a random variable having an F distribution with specified numerator and denominator degrees of freedom falls below a given value. Note that we specify the critical value $\theta \cdot F_{\alpha,r-1,(r-1)(s-1)}$ and use 1-PROBF to obtain the powers of F_{t1}.

The exact test derived for testing the hypothesis $H_o^{(1)}$: $\alpha_1 = \alpha_2 = \cdots = \alpha_r$, is of the form

$$F^*(H) = \frac{\underset{\sim}{\ell}{}_a^{*\prime}\left(R^{*\prime} G_X{}^*R^*\right)^{-1}\underset{\sim}{\ell}{}_a^*/(r-1)}{SSE^*/(r-1)(s-1)},$$

where $\underset{\sim}{\ell}{}_a^*$ and SSE^* are described in (3.102) and (3.105), respectively. Under $H_a^{(1)}$

$$F^*(H) \sim F'\left[r-1,\ (r-1)(s-1),\ \left(R^*\underset{\sim}{\alpha}-\underset{\sim}{m}^*\right)'\left(R^{*\prime} G_X{}^*R^*\right)^{-1}\left(R^*\underset{\sim}{\alpha}-\underset{\sim}{m}^*\right)/2\sigma^{*2}\right]$$

where $\sigma^{*2} = \sigma^2_{\alpha\beta} + \lambda^F_{\max}\sigma^2_e$.

The noncentrality parameter is determined by specifying a value for $\sigma^2_{\alpha\beta}$ (without loss of generality we set $\sigma^2_e = 1$) and values for the fixed effects under $H_a^{(1)}$: not $H_o^{(1)}$. Once again the PROBF function in SAS is used to obtain the powers of interest (note: the function PROBF allows us to specify noncentrality parameter values).

As mentioned above, we need to consider values for the fixed effects under $H_a^{(1)}$. Two vectors for $\underset{\sim}{\alpha}$ were considered. These were ordered according to their Euclidean norms as

$$\underset{\sim}{\alpha}_1^r = \begin{bmatrix} 0.5 \\ 0.75 \\ 1.0 \\ 1.25 \end{bmatrix} \quad ; \quad \underset{\sim}{\alpha}_2^r = \begin{bmatrix} 10 \\ 2 \\ 6 \\ 8 \end{bmatrix} \quad ; \qquad (4.33)$$

(Note: in Table 4.7 Designs 1, 2, 3 and 4 use the first two elements of the vectors in Equation 4.33 above.)

The last component of the power study concerns the designs to consider in the study. As with the 2-fold nested model, the sums of squares in the analysis of variance table for unbalanced two-way cross classification designs no longer have the distributional properties found in balanced designs. Since the degree of deviation from complete balance may affect the power, we considered designs with low and high degrees of imbalance.

Khuri (1986), proposed a measure of departure from complete balance for a two-way cross classification model given by

$$\phi_4 = \frac{1}{1+c_4^2}, \tag{4.34}$$

where

$$c_4^2 = \{\sum_{ij} \frac{[n_{ij} - n_{..}/(rs)]^2}{n_{..}/(rs)}\}/n_{..}\,.$$

Table 4.5. Designs Used in the 2-Way Cross Classification Power Study (ϕ_4 is a measure of departure from complete balance).

DESIGN 1

$N = 36$ $\phi_4 = 0.35$

$r=2;\ s=2;$

$n_{11}=30,\ n_{12}=3;\ \ n_{21}=2,\ n_{22}=1;$

DESIGN 2

$N = 36$ $\phi_4 = 0.96$

$r=2;\ s=2;$

$n_{11}=7,\ n_{12}=12;\ \ n_{21}=8,\ n_{22}=9;$

DESIGN 3

$N = 36$ $\phi_4 = 0.35$

$r=2;\ s=3;$

$n_{11}=23,\ n_{12}=9,\ n_{13}=1;\ \ n_{21}=1,\ n_{22}=1,\ n_{23}=1;$

DESIGN 4

$N = 36$ $\phi_4 = 0.96$

$r=2;\ s=3;$

$n_{11}=5,\ n_{12}=5,\ n_{13}=7;\ \ n_{21}=5,\ n_{22}=6,\ n_{23}=8;$

DESIGN 5

$N = 36$ $\phi_4 = 0.35$

$r=4;\ s=2;$

$n_{11}=20,\ n_{12}=6;\ \ n_{21}=2,\ n_{22}=1;\ n_{31}=1,\ n_{32}=4;\ n_{41}=1,\ n_{42}=1;$

DESIGN 6

$N = 36$ $\phi_4 = 0.96$

$r=4;\ s=2;$

$n_{11}=5,\ n_{12}=4;\ \ n_{21}=3,\ n_{22}=6;\ n_{31}=4,\ n_{32}=4;\ n_{41}=5,\ n_{42}=5;$

4.2.3 Results

Table 4.6. Power values for Testing the Random Main Effects in the Two-Way Cross Classification Mixed Model

				α = .05		α = .10	
Design	ϕ_4	$\alpha^2_{\alpha\beta}$	α^2_β	Approx.	Exact	Approx.	Exact
1 (2×2)	.35	1	.5	.368	.063	.433	.125
			6.0	.512	.138	.621	.266
		4	.5	.222	.055	.274	.110
			6.0	.281	.093	.368	.184
2 (2×2)	.96	1	.5	.237	.069	.339	.137
			6.0	.498	.168	.515	.316
		4	.5	.129	.056	.172	.111
			6.0	.193	.098	.286	.193
3 (2×3)	.35	1	.5	.334	.073	.405	.143
			6.0	.513	.269	.647	.438
		4	.5	.164	.059	.221	.118
			6.0	.252	.152	.363	.274
4 (2×3)	.96	1	.5	.068	.088	.141	.169
			6.0	.280	.367	.465	.550
		4	.5	.053	.061	.120	.121
			6.0	.149	.169	.253	.300
5 (4×2)	.35	1	.5	.192	.110	.255	.195
			6.0	.429	.443	.548	.560
		4	.5	.080	.075	.131	.141
			6.0	.122	.279	.329	.400
6 (4×2)	.96	1	.5	.125	.142	.217	.238
			6.0	.494	.527	.605	.634
		4	.5	.068	.079	.140	.147
			6.0	.265	.304	.398	.428

Table 4.7. Power values for Testing the Fixed Effects in the Two-Way Cross Classification Mixed Model.
($\underset{\sim}{g}^r$ = 1 corresponds to $\underset{\sim}{g}_1^r$,
$\underset{\sim}{g}^r$ = 2 corresponds to $\underset{\sim}{g}_2^r$(see 4.33); ϕ_4 is a measure of departure from complete balance.)

(I = type I SS, II = type II SS, E = Exact)

Design	ϕ_4	$\underset{\sim}{g}^r$	$\sigma^2_{\alpha\beta}$	σ^2_β	type	α = .05	α = .10
1 (2×2)	.35	1	1	.5	I	.003	.026
					II	.060	.097
					E	.051	.102
				6.0	I	.006	.037
					II	.063	.109
					E	.051	.102
			4	.5	I	.010	.047
					II	.067	.082
					E	.050	.100
				6.0	I	.014	.052
					II	.010	.098
					E	.050	.100
		2	1	.5	I	.018	.047
					II	.067	.113
					E	.374	.667
				6.0	I	.021	.052
					II	.077	.120
					E	.374	.667
			4	.5	I	.057	.072
					II	.099	.131
					E	.229	.437
				6.0	I	.081	.099
					II	.110	.130
					E	.229	.437

Table 4.7 - continued

(I = type I SS, II = type II SS, E = Exact)

Design	ϕ_4	$\underset{\sim}{\alpha}^r$	$\sigma^2_{\alpha\beta}$	σ^2_β	type	$\alpha = .05$	$\alpha = .10$
2 (2×2)	.96	1	1	.5	I	.007	.037
					II	.041	.083
					E	.051	.103
				6.0	I	.017	.057
					II	.047	.094
					E	.051	.103
			4	.5	I	.019	.060
					II	.040	.081
					E	.050	.101
				6.0	I	.026	.071
					II	.053	.101
					E	.050	.101
		2	1	.5	I	.026	.062
					II	.100	.125
					E	.445	.760
				6.0	I	.076	.113
					II	.158	.196
					E	.445	.760
			4	.5	I	.082	.134
					II	.159	.214
					E	.243	.462
				6.0	I	.113	.153
					II	.163	.238
					E	.243	.462
3 (2×3)	.35	1	1	.5	I	.002	.019
					II	.023	.065
					E	.051	.102
				6.0	I	.010	.041
					II	.032	.078
					E	.051	.102
			4	.5	I	.018	.056
					II	.039	.088
					E	.050	.101

Table 4.7 - continued.

(I = type I SS, II = type II SS, E = Exact)

Design	ϕ_4	$\mathscr{g}^r$	$\sigma^2_{\alpha\beta}$	σ^2_β	type	α = .05	α = .10
				6.0	I	.024	.067
					II	.042	.092
					E	.050	.101
		2	1	.5	I	.039	.061
					II	.132	.185
					E	.756	.936
				6.0	I	.054	.073
					II	.150	.198
					E	.756	.936
			4	.5	I	.061	.076
					II	.081	.193
					E	.448	.687
				6.0	I	.072	.081
					II	.192	.201
					E	.448	.687
4 (2×3)	.96	1	1	.5	I	.010	.042
					II	.037	.082
					E	.052	.104
				6.0	I	.016	.066
					II	.035	.083
					E	.052	.104
			4	.5	I	.026	.067
					II	.040	.083
					E	.051	.101
				6.0	I	.033	.075
					II	.051	.082
					E	.051	.101
		2	1	.5	I	.042	.071
					II	.076	.083
					E	.901	.989

Table 4.7 - continued.

(I = type I SS, II = type II SS, E = Exact)

Design	ϕ_4	$\underset{\sim}{\alpha}^r$	$\sigma^2_{\alpha\beta}$	σ^2_{β}	type	$\alpha = .05$	$\alpha = .10$
				6.0	I	.065	.082
					II	.081	.095
					E	.901	.989
			4	.5	I	.066	.085
					II	.081	.099
					E	.502	.744
				6.0	I	.073	.093
					II	.109	.162
					E	.502	.744
5 (4×2)	.35	1	1	.5	I	.005	.023
					II	.031	.074
					E	.057	.113
				6.0	I	.011	.040
					II	.036	.080
					E	.057	.113
			4	.5	I	.020	.059
					II	.038	.091
					E	.053	.105
				6.0	I	.025	.063
					II	.050	.105
					E	.053	.105
		2	1	.5	I	.063	.096
					II	.087	.099
					E	.717	.895
				6.0	I	.094	.114
					II	.101	.131
					E	.717	.895
			4	.5	I	.098	.238
					II	.133	.269
					E	.376	.585
				6.0	I	.196	.268
					II	.225	.312
					E	.376	.585

Table 4.7 - continued.

(I = type I SS, II = type II SS, E = Exact)

Design	ϕ_4	$\mathcal{B}^r$	$\sigma^2_{\alpha\beta}$	σ^2_β	type	α = .05	α = .10
6 (4×2)	.96	1	1	.5	I	.006	.024
					II	.028	.062
					E	.060	.119
				6.0	I	.012	.041
					II	.039	.084
					E	.060	.119
			4	.5	I	.023	.061
					II	.038	.081
					E	.053	.106
				6.0	I	.028	.066
					II	.042	.084
					E	.053	.106
		2	1	.5	I	.031	.046
					II	.096	.123
					E	.862	.969
				6.0	I	.054	.073
					II	.114	.191
					E	.862	.969
			4	.5	I	.123	.159
					II	.173	.217
					E	.421	.637
				6.0	I	.202	.261
					II	.298	.352
					E	.421	.637

4.2.4 Analysis of Power Study

Consider Table 4.6, where we have compared the power values for an approximate test and the derived exact test, used in testing the variance component associated with the random main effect. The power of the approximate test, for a fixed value of $\sigma^2_{\alpha\beta}$, appears to be an increasing function of σ^2_{β}. However, when we fix σ^2_{β}, the power for this test is a decreasing function of $\sigma^2_{\alpha\beta}$. When we hold constant the levels of the fixed main effects and the levels of the random main effects, and compare the designs only by changing the degree of imbalance (that is, different values for the n_{ij}'s), we find that except for Designs 5 and 6, the power of the approximate test is larger for the design with the higher degree of imbalance.

The power of the exact test behaves in a similar fashion. That is, if we hold $\sigma^2_{\alpha\beta}$ constant, the power is an increasing function of σ^2_{β}, and when we fixed σ^2_{β} the power is a decreasing function of $\sigma^2_{\alpha\beta}$. However, unlike the approximate test, the exact test performs better when the design has a lower degree of imbalance. In addition, note that when we hold the levels of the fixed effects constant, and change the number of levels for the random main effects, the power of the exact test is larger for those designs with more levels in the random main effects (compare Design 1 and 2 against Designs 3 and 4). A similar increase occurs when we hold the levels of the random main effects fix, and increase the number of levels of the fixed effects.

Note that, with the exception of Designs 5 and 6 (both 4×2), in general, the power of the approximate test does better than the power of the exact test. Unlike the exact test, in general the results from this table do not indicate an increase in the power of the approximate test when we increase the levels of either main effect (the random or fixed). Consequently, I recommend using the exact test when we have a larger number of levels for either the random or fixed main effect. In addition, since it appears that the approximate test is a decreasing function of the degree of imbalance, whereas, the exact test is an increasing function, I recommend using the exact test for those designs with a lower level of imbalance.

Consider Table 4.7 whose contents are the power values for the type I and type II approximate tests in addition to the power values for the exact test. We immediately notice the complete domination of the type II powers compared to the type I powers, however, it is still of interest to study the results of the type I test and compare them to the exact test.

We find that when we fix $\sigma^2_{\alpha\beta}$ the type I powers are an increasing function of the variance component σ^2_{β}. Similarily, when we hold σ^2_{β} constant the type I powers are an increasing function of $\sigma^2_{\alpha\beta}$. The relationship of the type II powers and the variance components is identical to the relationship we found for the type I powers.

Note that the powers of the exact test are independent of the variance component associated with the random main effect, that is, σ_β^2. As expected, the power of the exact test is a decreasing function of the variance component associated with the random interaction effect, $\sigma_{\alpha\beta}^2$. Comparison of Designs 1 and 5 or Designs 1 and 3 (for $\alpha = 2$), appears to indicate that an increase in the number of levels of either effect, increase the power of the exact test.

The exact test, in general, appears to have greater power than both the type I and type II approximate tests as one can observe for the designs in Table 4.7. However, the powers of the approximate tests are not very reliable because of the simulation employed. Hence, further studies with a better technique for obtaining the critical values of the approximate tests will need to be considered.

CHAPTER FIVE
FUTURE RESEARCH

The two models considered in CHAPTER THREE, the 2-fold nested mixed model and the two-way cross classification mixed model have been studied under certain assumptions that may not always hold true. In the derivations we assumed that a particular set of random effects had a variance-covariance structure which could be expressed as an unknown parameter (the variance component) times the identity matrix. It is possible that situations may arise where correlations exist within a particular set of random effects in which cases the identity matrix does not appear in the variance structure of the effects.

In addition, correlations may exist across two sets of random effects. We have assumed in the past, for instance, that the random main effect and random interaction effect in the two-way cross classification model are independently distributed. The derivations were based on this assumption. Again, it is possible, and indeed more realistic to assume that this assumption may be violated in certain situations.

A more pressing future research area is the determination of the eigenvectors (used in the frequent transformations in the process of arriving at a diagonal variance-covariance structure) that will provide optimal power values for the test.

Finally, generalizations of the proposed exact techniques to more complex models should be considered. Such generalizations in the case of nested models will not present any problems if there is only a single set of random effects with stages of higher hierachy being associated with fixed effects. However, most nested models involve only one set of fixed effects, occurring at the first stage, with the remaining effects having some type of random distribution. Once we consider more than one set of random effects in the nested models, the above derivations fail because we are not able to simultaneously diagonalize the matrices involved in the variance-covariance structure of the vector of observations.

Unlike the nested models, the two-way cross classification mixed model studied in CHAPTER THREE, had a variance-covariance structure involving two matrices that we could simultaneously diagonalize. This was necessary since the model had two sets of random effects. However, more complex cross classification models, such as the three-way cross classification model involving more than two sets of random effects, will lead to variance-covariance structures requiring the simultaneous diagonalization of more than two matrices. Obviously, we are left with numerous research problems to consider in the years to come.

APPENDIX A
THE EIGENVALUES OF THE MATRIX A_1 IN EQUATION 3.4

The matrix A_1, defined in equation (3.4), has the following eigenvalues:

"a" positive eigenvalues: $m_1, m_2, \ldots, m_a$

and

a zero eigenvalue of multiplicity $m_{\bullet}-a$,

where m_i is the number of levels of factor B nested within level i of factor A ($i = 1, 2, \ldots, a$).

Proof: From (Graybill 1984, p. 229, Theorem 8.8.16) if $\lambda_j^{(i)}$ for $j=1,2,\ldots,m_i$ are the eigenvalues of W_i for $i=1,2,\ldots,a$ (where W_i is an $m_i \times m_i$ matrix), then $\lambda_j^{(i)}$ for $j=1,2,\ldots,m_i$; $i=1,2,\ldots,a$ are the eigenvalues of $\bigoplus_{i=1}^{a} W_i$. We have $A_1 = B_1 B_1'$, where $B_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i}$. Hence, A_1 can be expressed as

$$A_1 = \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i} \right) \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i} \right)'$$

$$= \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i} \right) \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}'_{m_i} \right)$$

$$= \bigoplus_{i=1}^{a} \underset{\sim}{1}_{m_i} \underset{\sim}{1}'_{m_i} = \bigoplus_{i=1}^{a} J_{m_i}$$

Now, determine the eigenvalue of J_{m_i} for $i=1,2,\ldots,a$. That is, solve the characteristic equation:

$$|J_{m_i} - \lambda I_{m_i}| = 0, \text{ for } i=1,2,\ldots,a.$$

The determinant $|J_{m_i} - \lambda I_{m_i}|$ is equal to

$$|J_{m_i} - \lambda I_{m_i}| = \begin{vmatrix} 1-\lambda & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & 1-\lambda & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1-\lambda \end{vmatrix}$$

$$= \left[(1-\lambda) + (m_i - 1) \cdot 1\right](1-\lambda-1)^{m_i-1}$$

$$= (\lambda-m_i)\lambda^{m_i-1}$$

(see Graybill 1984, p.230, Theorem 8.9.1). This implies that the matrix J_{m_i} has eigenvalues, $\lambda = m_i$ with multiplicity equal to one, and $\lambda = 0$ with multiplicity equal to $m_i - 1$ for $i=1,2,\ldots,a$. Since, $A_1 = \bigoplus_{i=1}^{a} J_{m_i}$, it follows from the theorem described in the beginning of the proof, that A_1 is a matrix with eigenvalues m_1, m_2, ..., m_a each having multiplicity one, and zero having multiplicity equal to $m_{\bullet}-a$.

APPENDIX B

THE EIGENVECTORS ASSOCIATED WITH THE POSITIVE EIGENVALUES OF THE MATRIX A_1 OF EQUATION 3.4

The matrix A_1, defined in (3.4), has the following eigenvectors associated with the positive eigenvalues:

$$m_i^{-1/2} \underset{\sim}{1}^{(i)\prime} \quad \text{for } i=1,2,\ldots,a$$

where

$$\underset{\sim}{1}^{(1)\prime} = [\underset{\sim}{1}'_{m_1} : \underset{\sim}{0}_{1\times(m_\bullet - m_1)}]$$

$$\underset{\sim}{1}^{(2)\prime} = [\underset{\sim}{0}_{1\times m_1} : \underset{\sim}{1}'_{m_2} : \underset{\sim}{0}_{1\times(m_\bullet-(m_1+m_2))}]$$

$$\vdots$$

$$\underset{\sim}{1}^{(a)\prime} = [\underset{\sim}{0}_{1\times(m_\bullet - m_a)} : \underset{\sim}{1}'_{m_a}] .$$

Proof: Consider $A_1\underset{\sim}{1}^{(\ell)}$, where $\ell = 1, 2, \ldots, a$.

$$A_1\underset{\sim}{1}^{(\ell)} = \bigoplus_{i=1}^{a} J_{m_i}\underset{\sim}{1}^{(\ell)}$$

$$= \bigoplus_{i=1}^{a} J_{m_i} \left(\underset{\sim}{0}_{m_1} \oplus \underset{\sim}{0}_{m_2} \oplus \ldots \oplus \underset{\sim}{1}_{m_\ell} \ldots \oplus \underset{\sim}{0}_{m_a}\right) \underset{\sim}{1}_a$$

(where $\underset{\sim}{0}_{m_i}$ is an $m_i \times 1$ column vector of zeroes)

$$= \left(J_{m_i} \underset{\sim}{0}_{m_1} \oplus J_{m_2} \underset{\sim}{0}_{m_2} \oplus \ldots \oplus J_{m_\ell} \underset{\sim}{1}_{m_\ell} \oplus \ldots \oplus J_{m_a} \underset{\sim}{0}_{m_a}\right) \underset{\sim}{1}_a$$

$$= \left(\underset{\sim}{0}_{m_1} \oplus \ldots \oplus m_\ell \underset{\sim}{1}_{m_\ell} \oplus \ldots \oplus \underset{\sim}{0}_{m_a}\right) \underset{\sim}{1}_a$$

$$= m_\ell \underset{\sim}{1}^{(\ell)} \quad .$$

From Appendix A we know that m_ℓ is an eigenvalue of A_1; hence, $\underset{\sim}{1}^{(\ell)}$ must be an eigenvector of A_1 corresponding to the eigenvalue m_ℓ. Since, ℓ was arbitrarily chosen, it follows that $m_i^{-1/2} \underset{\sim}{1}^{(i)}$ for $i=1,2,\ldots,a$ are eigenvectors of A_1 associated with the eigenvalues m_i for $i=1,2,\ldots,a$, respectively.

APPENDIX C

THE EIGENVALUES OF THE MATRIX $C_tA_2C_t'$ IN EQUATION 3.68

The matrix $C_tA_2C_t'$ from (3.68) has eigenvalues r and 0 with multiplicity equal to s-1 and (r-1)(s-1), respectively.

Proof:

$$\begin{aligned} C_tA_2C_t' &= (I_r \times T)(J_r \times I_s)(I_r \times T') \\ &= (J_r \times T)(I_r \times T') \\ &= J_r \times I_{s-1}. \end{aligned}$$

We first obtain the eigenvalues of J_r, that is, we obtain the solution to the characteristic equation:

$$|J_r - \lambda I| = 0.$$

But, $|J_r = \lambda I| = (\lambda - r)\lambda^{r-1}$, from Graybill (1984, p.230, Theorem 8.9.1). Hence, J_r has an eigenvalue equal to r with multiplicity one and an eigenvalue equal to zero with multiplicity r-1. It follows from Graybill (1984, p.229, Theorem 8.8.16) that the matrix $C_t\alpha_2C_t' = J_r \times I_{s-1}$ has eigenvalues r and 0 with multiplicity equal to s-1 and (r-1)(s-1), respectively.

APPENDIX D

THE PRODUCT OF THE MATRIX T (3.66) AND ITS TRANSPOSE

The matrix

$$T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & \cdot\ \cdot & \cdot & \cdot & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{\sqrt{s(s-1)}} & \frac{1}{\sqrt{s(s-1}} & \cdot & \cdot & \cdot & \cdot & \frac{-(s-1)}{\sqrt{s(s-1)}} \end{bmatrix},$$

with (s-1) rows and s columns has the following property

$$T'T = I_s - \frac{1}{s} J_s .$$

Proof: By mathematical induction. Consider s=2 (that is, T has 1 row and 2 columns):

$$T'T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= I_2 - \frac{1}{2} J_2 \quad .$$

Now assume the property holds for $s = t$ and show that for $s = t+1$, $T'T = I_{t+1} - \frac{1}{(t+1)} J_{t+1}$. When $s = t+1$, T can be expressed as

$$T = \begin{bmatrix} T_t & \underset{\sim}{0}_{(t-1)\times 1} \\ \underset{\sim}{h}'_t & \frac{-t}{\sqrt{t(t+1)}} \end{bmatrix} ,$$

where

$$T_t = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & \cdot & \cdot & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{t(t+1)}} & \frac{1}{\sqrt{t(t+1)}} & \frac{1}{\sqrt{t(t+1)}} & \cdot & \cdot & \frac{-(t-1)}{\sqrt{t(t+1)}} \end{bmatrix}_{(t-1)\times t}$$

$$\underset{\sim}{h}_t' = \begin{bmatrix} \frac{1}{\sqrt{t(t+1)}} & \frac{1}{\sqrt{t(t+1)}} & \cdot\ \cdot\ \cdot & \frac{1}{\sqrt{t(t+1)}} \end{bmatrix}_{1\times t} .$$

Note by our assumption that the property holds for s=t,

$T_t'T_t = I_t - \frac{1}{t} J_t$.

Hence,

$$T'T = \begin{bmatrix} T_t'T_t + \underset{\sim}{h}_t'\underset{\sim}{h}_t & \frac{-t}{\sqrt{t(t+1)}} \underset{\sim}{h}_t' \\ \frac{-t}{\sqrt{t(t+1)}} \underset{\sim}{h}_t & \frac{t^2}{t(t+1)} \end{bmatrix}$$

$$= \begin{bmatrix} I_t - \frac{1}{t} J_t + \frac{1}{t(t+1)} J_t & \frac{-1}{(t+1)} \underset{\sim}{1}_t \\ \frac{-1}{(t+1)} \underset{\sim}{1}_t' & \frac{t}{t+1} \end{bmatrix}$$

$$= \quad I_{t+1} - \begin{bmatrix} \frac{1}{t} J_t - \frac{1}{t(t+1)} J_t & \frac{1}{(t+1)} \underset{\sim}{1}_t \\ \frac{1}{(t+1)} \underset{\sim}{1}_t' & \frac{1}{t+1} \end{bmatrix}$$

$$= I_{t+1} - \frac{1}{(t+1)} J_{t+1} .$$

Hence, by induction, in general for $s > 2$

$$T'T = I_s - \frac{1}{s} J_s.$$

APPENDIX E

COMPUTER PROGRAM TO OBTAIN EXACT TESTS FOR THE RANDOM AND FIXED EFFECTS IN AN UNBALANCED 2-FOLD NESTED MIXED MODEL

```
*****************************************************************

*   NOTE THAT THE MACRO HAS BEEN STRUCTURED TO ALLOW A MAXIMUM   *

*            OF 6 LEVELS FOR THE NESTING FIXED EFFECT            *

*            AND 3 LEVELS FOR NESTED RANDOM EFFECT               *

*                                                                *

*     ASSUMPTION: N > 2B    (N = Σn_ij,   B = Σm_i)              *

*****************************************************************

MACRO EXACT (R11,R12,R13,R21,R22,R23,R31,R32,R33,R41,R42,R43,R51,

             R52,R53,R61,R62,R63,M1,M2,M3,M4,M5,M6,N,RRA);

PROC MATRIX;

FETCH Y DATA = Z;

    %IF %LENGTH(&R11) = 0 %THEN %GOTO LVN11;

        M1     =  &M1;

        N1     =  &R11;

        RIN11  =  1#/N11;

        V      =  1#/N11;

        RXX11  = J.(N11,N11,RIN11);

        GG     =  RXX11;

        VM1    =  J.(&M1,1,1);
```

```
          VV          =   BLOCK(VM1);

          B           =   &M1;

          S           =   M1-1;

          YVI         =   &SUM(Y(1:N11,))#/N11;

          IF M1       =   3 THEN DO;

             C11      =   1#/(SQRT(2));

             C12      =   -1#/(SQRT(2));

             C13      =   0;

             C21      =   1#/(SQRT(6));

             C22      =   1#/(SQRT(6));

             C23      =   -2#/(SQRT(6));

             CONT     =   (C11||C12||C13)//(C21||C22||C23);END;


          IF M1       =   2 THEN DO;

             C11      =   1#/(SQRT(2));

             C12      =   -1#/(SQRT(2));

             CONT     =   C11||C12;END;

LVN11:    %IF LENGTH(&R12)=0 %THEN %GOTO LVN12;


          N12         =   &R12;

          RIN12       =   1#/N12;

          RXX12       =   J.(N12,N12,RIN12);

          GG          =   BLOCK(GG,RXX12);

          V           =   V||RIN12;

          SS          =   N11+N12;
```

```
                 .


                 .


                 .


LVN62: %IF %LENGTH(&R63)=0 %THEN %GOTO LVN63;
        N63          =  &R63;
        RIN63        =  1#/N63;
        RXX63        =  J.(N63,N63,RIN63);
        GG           =  BLOCK(GG,RXX63);
        V            =  V||RIN63;
LVN63:  YVI          = YVI//(SUM(Y(SS+1:SS+N63,)))#/N63;

        GG1          =  GG;
        B1           =  VV;
        WS           =  &N;
        A            = &RRA;UNUM = A-1;
        I1           =  I(B);
        I2           =  I(WS);
        I5           = I(S);
        R1           = S;
        NT2          = WS-B;
        N2           = B-A;
        OB          = J.(B,1,1);

K = DIAG(V);


*****************************************************************
*  I.  AN EXACT TEST FOR THE VARIANCE COMPONENT ASSOCIATED WITH *
*      THE RANDOM NESTED EFFECT                                 *
*****************************************************************;


Z         =  CONT*YVI;

K1        =  CONT*K*CONT %STR(%');

QB        =  Z %STR(%')*INV(K1)*Z;


CW        =  I2-GG1;

SSE       =  Y %STR(%')*CW*Y;


*****************************************************************;

        FB      =  (QB#/R1)#/(SSE#/NT2);



    *********************************************************

    *     II.  AN EXACT TEST FOR THE FIXED EFFECTS          *

    *********************************************************;


LAMAX           =  MAX(EIGVAL(K));

EIGEN           E  HH1 CW;
```

```
HH1            =  HH1 %STR(%');

H1             =  HH1(1:B,);

MAT1           =  (LAMAX*I1 - K);

EIGEN          MM1 MM2 MAT1;

  DO I         = 1 TO B;

                  IF MM1(I,1) < .00001 AND MM1 (I,1) > -.00001 THEN

                     MM1(I,1) = SQRT(MM1(I,1));

MM1            =  DIAG(MM1);

MAT2           =  MM2*MM1*MM2 %STR(%');

               =  YVI + MAT2*H1*Y;


     ********************************************************

     *   OBTAIN A SOLUTION FOR THE VECTOR OF FIXED EFFECTS  *

     *   USING THE NORMALIZED EQUATIONS                     *

     ********************************************************;

               =  OB||B1;

GXPX                =  GINV(X %STR(%')*X);

ES             =  GXPX*X %STR(%')*W;


O1             =  J.(UNUM,1,0);

R = 1 -1 0 0/

    0 1 -1 0/

    0 0 1 -1;

R = R(1:A-1,1:A);

R = O1||R;
```

```
LA = R*ES;

     ****************************************************

     *            NUMERATOR OF THE TEST STATISTIC          *

     ****************************************************;

QA              =  (LA %STR(%')* INV(R*GXPX*R %STR(%'))*LA)#/UNUM;

     ****************************************************

     *          DENOMINATOR OF THE TEST STATISTIC          *

     ****************************************************;

SSE2            =  (W %STR(%')*(I1 - X*GXPX*X %STR(%'))*W)#/N2;

     ****************************************************

     *             TEST STATISTIC   FH                     *

     ****************************************************;

FH              =  QA#/SSE2;

MAT             =  FB||R1||NT2||FH||UNUM||N2;

OUTPUT MAT OUT  =  NEW(RENAME  =  (COL1  = FB COL2  = R1

                                    COL3  =  NT2 COL4  =  FH

                                    COL5  =  UNUM  COL6  =  N2));

DATA NEW;SET NEW;
```

```
        PV1 = 1 - PROBF(FB,R1,NT2);

        PV2 = 1 - PROBF(FH,UNUM,N2);

PROC PRINT; VAR FB R1 NT2 PV1 FH UNUM N2 PV2;

******************************************************************

*       PV1 = power for the exact test of random effects         *

*       PV2 = power for the exact test of fixed effects.         *

******************************************************************;
```

APPENDIX F

COMPUTER PROGRAM TO OBTAIN AN EXACT TEST FOR THE RANDOM AND FIXED EFFECTS IN AN UNBALANCED TWO-WAY CROSS CLASSIFICATION MIXED MODEL

```
*****************************************************************

*  NOTE:   THE MACRO HAS BEEN STRUCTURED TO ALLOW A MAXIMUM OF   *

*            - 4 LEVELS FOR THE FIXED MAIN EFFECT - A             *

*            - 4 LEVELS FOR THE RANDOM MAIN EFFECT - B            *

*                                                                 *

*  ASSUMPTION: N > MAX(2AB-A,2AB-B)    (N = Σn_ij)                *

*****************************************************************;

OPTIONS NONOTES;

%MACRO EXACT2(N11,N12,N13,N14,N21,N22,N23,N24,N31,N32,N33,N34,N41,

              N42,N43,N44,N,A,B);

PROC MATRIX;

FETCH Y DATA = A;

    N = &N;

    A = &A; B = &B; G = (A-1);

   RS = A*B;

   N1 = (B-1);

   D1 = (A-1)*(B-1);

   N2 = D1;
   D2 = N-2*A*B + A;
          %IF %LENGTH(&N11) = 0 %THEN %GOTO LVN12;
          N11   = &N11;
          X11   = 1#/N11;
          K     = X11;
          JJ    = J.(N11,N11,X11);
          YVAR = (SUM(Y(1:N11,)))#/N11;
%LVN12:   %IF %LENGTH(&N12) = 0 %THEN %GOTO LVN13;
          N12   = &N12;
          X12   = 1#/N12;
          K     = BLOCK(K,X12);
          JJ    = BLOCK(JJ,J.(N12,N12,X12));
          SS    = N11+N12;
          YVAR = YVAR//(SUM(Y(N11+1:SS,)))#/N12;
                .
                .
                .
%LVN44:   %IF %LENGTH(&N44) = 0 %THEN %GOTO LVN51;
          N44   = &N44;
          X44   = 1#/N44;
          K     = BLOCK(K,X44);
          JJ    = BLOCK(JJ,J.(N44,N44,X44));
          YVAR = YVAR//(SUM(Y(SS+1:SS+N44,)))#/N44;
```

```
     *******************************************************
     *   I.  A TEST FOR THE FIXED EFFECTS OF THE MODEL     *
     *******************************************************;

   *********************************************************
   *   DETERMINE THE ORTHOGONAL MATRIX THAT SIMULTANEOUSLY *
   *   DIAGONOLIZES BOTH A1 AND A2 (WHERE A1 AND A2 ARE    *
   *   ASSOCIATED WITH THE FIXED AND RANDOM MAIN EFFECTS,  *
   *   RESPECTIVELY)                                       *
   *                A1 = B1*B1'                            *
   *                A2 = B2*B2'                            *
   *   DENOTE THIS MATRIX BY P                             *
   *********************************************************;
%LVN51:
        A1       = I(A)@(J(B,B,1));
        A2       = (J(A,A,1))@I(B);
        EIGEN M1 P1 A1;
        P2       = P1 %STR(%');
        C        = P2*A2*P2 %STR(%');
        EIGEN M2 P3 C;
        P        = P1*P3;
        P        = P(2:RS,);
        C1       = P*A1*P %STR(%');
        C2       = P*A2*P %STR(%');
```

```
          DO       I = 1 TO RS-1;
                   IF C1(I,I) < .00001 AND C1(I,I) > -.00001
THEN                                                        C1(I,I) = 0;
                   IF C2(I,I) < .00001 AND C2(I,I) > -.00001
THEN                                                        C2(I,I) = 0;
                   END;
          P2 = P(LOC(VECDIAG(C1)>0),);
          C3 = C1 + C2;
          DK = VECDIAG(C3);


          DO       I = 1 TO RS-1;
                   IF DK(I,) = 0 THEN DO;
                               DG = DG||I;
                               END;
                               END;


          P4 = P(DG,);
          P24 = P2//P4;


          B1       = I(A)@J(B,1,1);


      *****************************************************
      *            OBTAIN THE TRANSFORMATION (3.93)        *
      *****************************************************;
L24     = P24*K*P24 %STR(%');
```

```
LAM1      = MAX(EIGVAL(L24));

NU1       = B*(A-1);

IV1       = I(NU1);

MAT1      = (LAM1*IV1 - L24);

EIGEN M1 M2 MAT1;

DO I      = 1 TO NU1;

   IF M1(I,1) < .00001 AND M1(I,1) > -.00001 THEN M1(I,1) = 0;

      M1(I,1) = SQRT(M1(I,M1));

      END;

MM1       = DIAG(M1);

MAT2      = M2*MM1*M2 %STR(%');

W         =(I(N) - JJ);

EIGEN WW1 WW2 W;

HF1       = WW2(,1:NU1);

W1        = P24*YVAR + MAT2*HF1 % STR(%') * Y;


     ****************************************************

     *  OBTAIN A SOLUTION FOR THE VECTOR OF FIXED EFFECTS  *

     *  USING THE NORMALIZED EQUATIONS                     *

     ****************************************************;


   X      = P24*B1;

   GX     = GINV(X %STR(%')* X);

GHAT      = GX*X %STR(%') * W1;

   R      = 1 -1 0 0 /
```

```
          0 1 -1 0 /
          0 0 1 -1;
   R    = R(1:A-1,1:A);

      ****************************************************
      *  NON1 = NUMERATOR OF THE F STATISTIC             *
      ****************************************************;

 NON1 = ((R*GHAT) %STR(%')*INV(R*GX*R %STR(%'))*(R*GHAT))#/G;

      ****************************************************
      *  DETERMINE THE ERROR SUM OF SQUARES OF THE MODEL  *
      ****************************************************;

 SSE = W1 %STR(%')*W1 - (GHAT %STR(%') * X %STR(%') *W1);

      ****************************************************
      *  TO TEST THE FIXED EFFECTS OF THE MODEL WE USE THE  *
      *  TEST STATISTIC FH  (3.106)                       *
      ****************************************************;

 FH = NON1#/(SSE#/N2);
      *****************************************************
      *  II.  A TEST FOR THE RANDOM MAIN EFFECT OF THE MODEL *
      *****************************************************;
```

```
T1        = (1#/SQRT(2)||-1#/SQRT(2)||0||0);

T2        = (1#/SQRT(6)||1#/SQRT(6)||-2#/SQRT(6)||0);

T3        = (1#/SQRT(12)||1#/SQRT(12)||1#/SQRT(12)||-3#/SQRT(12));

T4        = T1//T2//T3;

T         = T4(1:B-1,1:B);

CT        = I(A)@T;


     ****************************************************

     *  OBTAIN THE TRANSFORMATION (3.79)                *

     ****************************************************;

ZT        = CT*YVAR;


CTA2      = CT*A2*CT %STR(%');

EIGEN MM PT CTA2;

UT        = PT %STR(%') * ZT;


NU2       = A*(B-1);

HT        = WW2(,1:NU2);


LT        = PT %STR(%') * CT*K* CT %STR(%') * PT;

L2        = MAX(EIGVAL(LT));

MAT3      = (L2*I(NU2) - LT);

EIGEN M1 M2 MAT3;

DO I      = 1 TO NU2;

   IF M1(I,1) < .00001 AND M1(I,1) > -.00001 THEN M1(I,1) = 0;
```

```
      M1(I,1) = SQRT(M1(I,1));
      END;
MM1     = DIAG(M1);
MAT4    = M2*MM1*M2 %STR(%');
WT      = UT + MAT4*HT %STR(%') * Y;

     *****************************************************
     *  OBTAIN THE NUMERATOR OF THE TEST STATISTIC       *
     *****************************************************;

WT1     = WT(1:B-1);
QTB     = (WT1 %STR(%') * WT1)#/N1;

     *****************************************************
     *  OBTAIN THE DENOMINATOR OF THE TEST STATISTIC     *
     *****************************************************;

WT2     = WT(1:B,);
QTAB    = (WT2 %STR(%') * WT2)#/D1;

     *****************************************************
     *   TO TEST THE FIXED EFFECTS OF THE MODEL WE USE   *
     *   THE TEST STATISTIC FT1 (3.84)                   *
     *****************************************************;
FT1     = QTB#/QTAB;
```

```
        ************************************************
        *   III.  A TEST FOR THE RANDOM INTERACTION EFFECT OF *
        *          THE MODEL                                   *
        ************************************************;
HT2      = WW2(,NU2+1:N-RS);
QT2      = (Y %STR(%')*HT2*HT2 %STR(%')*Y)#/D2;


FT2      = QTAB#/QT2;


ALL      = FT2||D1||D2||FT1||N1||FH||G||N2;
OUTPUT ALL OUT = NEW(RENAME = (COL1 = FT2 COL2 = D1 COL3 = D2
                               COL4 = FT1 COL5 = N1 COL6 = FH
                               COL7 = G   COL8 = N2));
DATA ALL;
      SET NEW;
      PFT2 = 1 - PROBF(FT2,D1,D2);
      PFT1 = 1 - PROBF(FT1,N1,D1);
      PFH  = 1 - PROBF(FH,G,N2);
PROC PRINT;
      VAR FT2 D1 D2 PFT2 FT1 N1 PFT1 FH G N2 PFH;
***********************************************************
*  PFT2 = power for the Exact Test of the Interaction Effect.   *
*  PFT1 = power for the Exact Test of the Random Main Effect.   *
*  PFH  = power for the Exact Test of the Fixed Main Effect.    *
***********************************************************;
```

APPENDIX G
IMHOF COMPUTER PROGRAM

```
*****************************************************************
*   PROCEDURE FOR COMPUTING THE POWERS OF THE APPROXIMATE TEST   *
*     1.  OBTAIN CRITICAL VALUES FROM MONTE CARLO SIMULATION     *
*     2.  USING THESE CRITICAL VALUES WE APPLY THE PROGRAM       *
*         IN THIS APPENDIX (SEE KOERTS AND ABRAHAMSE (1969)).    *
*  D = VECTOR CONTAINING THE POSITIVE EIGENVALUES OF THE MATRIX  *
*      ASSOCIATED WITH THE QUADRATIC FORM INVOLVED IN COMPUTING  *
*      THE POWER.                                                *
*                                                                *
*  MULT = VECTOR CONTAINING THE MULTIPLICITIES OF THE ABOVE      *
*         EIGENVALUES.                                           *
*                                                                *
*  DELTA = VECTOR CONTAINING THE CORRESPONDING NONCENTRALITY     *
*          PARAMETERS.                                           *
****************************************************************;


PROC MATRIX;
FETCH MAT DATA = NEW2(KEEP = D DELTA MULT);
NR = NROW(MAT);
D = MAT(,1);
```

```
DELTA = MAT(,2);

MULT = MAT(,3);

XKRIT = 0;

EPS1 = .00001;EPS2 = .00001;

XK = 0;SLAM = 0; SDELT = 0;

DO I = 1 TO NR;

   XK = XK + 0.5*MULT(I,);

   SLAM = SLAM + 0.5*MULT(I,)*LOG(ABS(D(I,)));

   SDELT = SDELT + 0.5*DELTA(I,);

   END;


UB = EXP(-(LOG(EPS1*XK)+1.14472989+(SLAM+SDELT))#/XK);

LL1:SUM = 0;

DO I = 1 TO NR;

   DD = (D(I,)*UB)**2;

   SUM = SUM + DELTA(I,)*DD#/(1+DD);

   END;


SUM = 0.5*SUM;

TU = EXP(1.3372989+LOG(XK)+(SLAM+SUM)+XK*LOG(UB));

CH1 = 1#/TU-EPS1;

IF CH1 <= THEN GOTO LL20;

UB = UB + 5#/XK;

GOTO LL1;
```

```
LL10:U = U;
   IF U <= 0 THEN GOTO LL15;
   TETA = 0;
   RHO = 1;
   SUM = 0;
   DO I = 1 TO NR;
       C1 = D(I,)*U;
       C2 = 1+C1**2;
       TETA = TETA+MULT(I,)*ATAN(C1)+DELTA(I,)*C1#/C2;
       SUM = SUM + DELTA(I,)*C1**2#/C2;
       RHO = RHO*(C2##(.25*MULT(I,)));
       END;
TETA = 0.5*(TETA-XKRIT*U);
RHO = RHO*EXP(0.5*SUM);
FBL = (SIN(TETA))#/(RHO*U);
GOTO LL18;
LL15:   FBL = 0;
        DO I = 1 TO NR;
           FBL = FBL + D(I,)*(MULT(I,)+DELTA(I,));
        END;
FBL = 0.5*(FBL-XKRIT);
LL18:  IF KSKIP = 1 THEN GOTO LL21;
       IF KSKIP = 2 THEN GOTO LL22;
       IF KSKIP = 3 THEN GOTO LL23;
       IF KSKIP = 4 THEN GOTO LL24;
```

```
          IF KSKIP = 5 THEN GOTO LL25;
LL21:     OU = 1;RETURN;
LL22:     OU = 2;RETURN;
LL23:     OU = 3;RETURN;
LL24:     OU = 4;RETURN;
LL25:     OU = 5;RETURN;
LL20:     RANGE = UB;
          MH = 1;
          U = RANGE*0.5;
          KSKIP = 1;
          LINK LL10;
          SUMK = FBL*RANGE*2#/3;
          U = 0;
          KSKIP = 2;
          LINK LL10;
          VINT2 = SUMK+FBL*RANGE#/6;
          U = UB;
          KSKIP = 3;
          LINK LL10;
          VINT2 = VINT2+FBL*RANGE#/6;
          FD = 0.5 + (-.3183*VINT2);
          DO NIT = 1 TO 14;
              FD1 = FD;
              VINT2 = (VINT2+(-SUMK*0.5))*0.5;
              MH = 2*MH;
```

```
            STEP = RANGE#/MH;

            U = STEP*0.5;

            KSKIP = 4;

            LINK LL10;

            SUMK = FBL;

            DO K = 2 TO MH;

                U = U + STEP;

                KSKIP = 5;

                LINK LL10;

                SUMK = SUMK + FBL;

                END;

                SUMK = SUMK*STEP*2#/3;

                VINT2 = VINT2 + SUMK;

                FD = 0.5+(-.318309886*VINT2);

                HH = (NIT - 3);

                IF HH <= 0 THEN GOTO LL28;

                FF = (ABS(FD1-FD)-EPS2);

                IF FF <0 THEN GOTO LL29;

LL28:           TT = 0;

                END;

LL29:SS = 0;

       OUTPUT FD OUT = NEW4(RENAME=(COL1 = TYPE1));

*******************************************************************

*                  1 - FD = POWER OF THE TEST.                    *

*******************************************************************;
```

BIBLIOGRAPHY

Cummings, W.B. and Gaylor, D.W. (1974). Variance component testing in unbalanced nested designs. Journal of the American Statistical Association 69, 765-771.

Golub, G.H. (1969). Matrix decompositions and statistical calculations, Statistical Computation, R.C. Milton and J.A. Nelder, eds., New York: Academic Press.

Graybill, F.A. (1983). Matrices with Applications in Statistics, 2nd Edition. Wadsworth International Group, Belmont, California.

Harville, D.A. and Fenech, A.P. (1986). Confidence intervals for a variance ratio, or for heritability, in an unbalanced mixed linear model. Biometrics, 41, 137-152.

Imhof, J.P. (1961). Computing the distribution of quadratic forms in normal variables. Biometrika, 48, 3 and 4, 419.

Khuri, A.I. (1981). Simultaneous confidence intervals for functions of variance components in random models. Journal of the American Statistical Association, 76, 878-885.

Khuri, A.I. (1982). Direct Products: A powerful tool for the analysis of balanced data. Communications in Statistics: Theory and Method, 11(25), 2903-2920.

Khuri, A.I. (1986). Measures of imbalance of unbalanced models. Technical Report Number 255, Department of Statistics, University of Florida, Gainesville, FL.

Khuri, A.I. (1986a). An exact test for the nesting effect's variance component on an unbalanced random two-fold nested model. Technical Report Number 248, Department of Statistics, University of Florida, Gainesville, FL. Soon to appear in Statistics and Probability Letters.

Khuri, A.I. and Littell, R.C. (1986b). Exact tests for the main effects' variance components in an unbalanced random two-way model. Technical Report Number 268, Department of Statistics, University of Florida, Gainesville, FL. Soon to appear in Biometrics.

Koerts, J. and Abrahamse, A.P.J. (1969). On the Theory and Application of the General Linear Model, 1st Edition. Rotterdam, Netherlands, University Press.

Lindgren, B.W. (1976). Statistical Theory, 3rd Edition. Macmillan Publishing Co., Inc., New York.

Miliken, G.A. and Johnson, D.E. (1984). Analysis of Messy Data. Volume 1: Designed Experiments, 1st Edition. Van Nostrand Reinhold Co., New York.

Montgomery, D.C. (1976). Design and Analysis of Experiments, 1st Edition. John Wiley & Sons, New York.

Satterthwaite, F.E. (1946). An approximate distribution of estimates of variance components. Biometric Bulletin, 2, 110-114.

Searle, S.R. (1971). Linear Models, 1st Edition, John Wiley & Sons, New York.

Searle, S.R. (1982). Matrix Algebra Useful for Statistics. John Wiley & Sons, New York.

Seely, J.F. and El-Bassiouni, Y. (1983). Applying Wald's variance component test. Annals of Statistics, 11, 197-201.

Spjϕtvoll, E. (1968). Confidence interval and tests for variance ratios in unbalanced variance components models. Review of the International Statistical Institute, 36, 1.

Statistical Analysis System Institute (1985). SAS User's Guide: Basics, 5th Edition. SAS INSTITUTE Inc., Cary, North Carolina.

Thomsen, I. (1975). Testing hypotheses in unbalanced variance components models for two-way layouts. Annals of Statistics, 3, 257-265.

Tietjen, G.L. (1974). Exact and approximate tests for unbalanced random effects designs. Biometrics, 30, 573-581.

Wald, A. (1940). On the analysis of variance with unequal class frequencies. Annals of Mathematical Statistics, 11, 96-100.

Wald, A. (1941). On the analysis of variance in case of multiple classifications with unequal class frequencies. Annals of Mathematical Statistics, 21, 346-350.

Wald, A. (1947). A note on regression analysis. Annals of Mathematical Statistics, 18, 586-588.

BIOGRAPHICAL SKETCH

I was born in Pinar Del Rio, Cuba on November 17, 1957, and immigrated to the United States in March of 1962. I lived in Miami, Florida from 1962 till graduation from high school in 1976. Since September of 1976 I have resided in Gainesville, Florida. I am married to the former Betty Ramos and we have two children, Gabriel and Natalia.

I received my B.S. degree in mathematics in May 1980 and in August 1972 I was awarded an M.A. degree in statistics, both from the University of Florida. As a graduate student in the department of statistics, I was employed at the Center of Instructional Research and Computing Activities (CIRCA) from July 1981 through July 1986. While in CIRCA, I worked under the direction of Drs. Michael Conlon and John Dixon, as a statistical consultant to graduate students, faculty, and research scientists.

At the present time, I am employed at Shands Teaching Hospital working as a statistician in the Information Services Department.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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